Applying Negishi's method to stochastic models with overlapping generations*

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May 30, 2014

Abstract

In this paper we develop a Negishi approach to characterizing recursive equilibria in stochastic models with overlapping generations. When competitive equilibria are Pareto-optimal, using *recursive Negishi weights* as a state variable has several major computational advantages over the standard approach of using financial wealth or portfolios. We show that the Negishi approach extends naturally to models with borrowing constraints where the welfare theorems fail. Moreover, we derive two sets of sufficient conditions for the existence of recursive equilibria. The first involves a strong gross-substitutes assumption on preferences, yet only weak assumptions on the Markov process that governs fundamentals. The second includes only weak assumptions on preferences, but requires that the exogenous shock contains two components with atomless distributions, a purely transitory shock to discounting as well as a shock to endowments and/or dividends which does not depend on last period's shocks directly.

*We thank seminar participants at various universities and conferences and in particular Hal Cole, Piero Gottardi, Dominika Kryczka, David Levine, Ramon Marimon, Thomas Mariotti, Stavros Panageas, Herakles Polemarchakis, Victor Rios-Rull, Yeneng Sun, and Gianluca Violante for helpful comments. We gratefully acknowledge financial support from the ERC.

1 Introduction

In infinite horizon exchange economies with overlapping generations, complete financial markets, and a Lucas tree, the welfare theorems hold and Negishi's (1960) approach to characterizing equilibrium allocations as the solution to a social planner's problem can be employed (see Kehoe et al. (1992)). In this paper we show how to formulate the equilibrium of a stationary stochastic overlapping generations (OLG) economy recursively using *recursive Negishi weights* as an endogenous state variable. Negishi's approach is typically considered useful only when the number of commodities is larger than the number of agents, and in the OLG model both are infinite. However, we show that using recursive Negishi weights to compute equilibria in OLG models can nevertheless result in large efficiency gains compared to conventional methods that approximate recursive equilibria on a natural state space such as agents' beginning-of-period portfolios or financial wealth. We also show how our recursive method can still be applied in the presence of borrowing constraints when the welfare theorems do not hold.

We analyze an OLG exchange economy with L perishable commodities and Markovian fundamentals. Each period H agents enter the economy, they live for A periods and maximize time-separable, expected utility. We first consider a model with complete financial markets where equilibrium allocations are Pareto-efficient and can be obtained as a solution to a planner's problem that maximizes the sum of all agents' utilities weighted by Pareto weights which ensure that the budgets of all agents are balanced. At each node of the event tree we define *instantaneous* Negishi weights as the Pareto weights of the agents currently alive, discounted by their respective discount rates since birth. Since utility is time- and state-separable, individuals' consumption at each node is a simple function of these weights. While the weights for the non-newborn agents are directly obtained by discounting their previous-period weights, determining the weights of the newborn agents is non-trivial. By writing the budget equations recursively we obtain a functional equation that determines these weights as a function of the exogenous shock and the weights of the non-newborn agents. As a second step we show how our approach extends naturally to debt-constrained models in which the welfare theorems fail. In this case, whenever borrowing constraints bind, instantaneous Negishi weights are no longer simply obtained by discounting the previous-period weights. It is natural to define the endogenous state variable to consist of the discounted previous-period instantaneous Negishi weights of the currently alive, which we call *recursive* weights. Without borrowing constraints, these recursive weights are of course equal to the current instantaneous Negishi weights of the non-newborn agents. In contrast, in the presence of debt constraints, instantaneous Negishi weights differ from the recursive weights whenever borrowing constraints bind. We argue that, despite this complication, it is still advantageous to use recursive weights as the endogenous state. Obviously, they can no longer be interpreted as discounted Pareto weights for a social planner's problem, yet together with the exogenous shock they are still a sufficient statistic for the current state of the economy. Using these recursive weights as the endogenous state we prove the existence of recursive equilibria under two sets of sufficient conditions. The first includes a strong gross-substitutes assumption on preferences, yet only weak assumptions on the Markov process that governs fundamentals. The second includes only weak assumptions on preferences, but stronger assumptions on the exogenous shocks and it also requires that households cannot borrow against future labor endowments.

Models with overlapping generations have widespread applications in public finance, macroeconomics, and finance (see e.g., Kotlikoff and Auerbach (1983) or Storesletten et al. (2007)). However, in the presence of aggregate risk the computation of equilibria in these models becomes very difficult even when financial markets are complete (see e.g. Rios-Rull (1996)). It turns out that many of these difficulties can be mitigated if recursive Negishi weights are used as an endogenous state variable. For models with infinitely lived agents, there are already various papers that use individuals' consumption or recursive Negishi weights as the endogenous state variable to facilitate the computation of equilibria. When markets are complete and equilibria Pareto-efficient, using recursive weights as the endogenous state has been shown to be useful when agents have heterogeneous beliefs and/or non-separable utility (see e.g., Beker and Espino (2011) or Borovička (2013)). In such models the welfare theorems hold and agents' consumption is characterized by fixed Negishi weights. Non-separable utility or differences in beliefs imply that recursive weights change with discounting and belief differences, but existence of recursive equilibrium is typically not a major problem. Following Marcet and Marimon (1999) many studies have examined models in which allocations are constrained-efficient and can be characterized as the solution to a programming problem with forward looking constraints (see e.g. Ljungqvist and Sargent (2004) for an overview). Messner et al. (2013) provide general conditions for existence of recursive equilibria in this framework. Several authors apply the method to models with infinitely lived agents, incomplete markets, and/or borrowing constraints (e.g., Cuoco and He (1994), Chien and Lustig (2010), Chien et al. (2011), Dumas and Lyasoff (2012), and Gottardi and Kubler (2013)). In these models allocations are not constrained-efficient and, with the exception of Gottardi and Kubler (2013), there are no results on the existence of recursive equilibria. Garleanu and Panageas (2014) consider a continuous-time, perpetual-youth model and by using recursive Negishi weights as the endogenous state variable they reduce the model dynamics to a simple system of differential equations. We show, for stochastic models with overlapping generations and borrowing constraints, that using recursive Negishi weights has computational and theoretical advantages over using the natural state.

The computational advantages of recursive Negishi weights are as follows: First, one needs to approximate much fewer functions to characterize the equilibrium dynamics of the economy. For the case of complete markets and no borrowing constraints, one needs only the map from the current state to instantaneous weights of the newborn agents. These are ZH functions, where Z

is the number of exogenous shocks and H is the number of agents per generation. In contrast, $H(A-1)Z^2$ functions are necessary to approximate the map from current financial wealth across agents to their financial wealth next period for each combination of shocks in the current and the subsequent period. Thus, the Negishi approach reduces the number of functions needed to capture equilibrium dynamics $(A-1)Z$ times. Importantly, the size of the systems of non-linear equations that have to be solved also decreases by the same factor. This reduces computation time substantially, if A or Z is large. Also, by using Negishi's method the computational burden barely increases with the number of physical commodities, while it increases substantially if the natural state space is used. Second, the use of recursive Negishi weights as states allows for a straightforward error analysis. Approximation errors can be interpreted as transfers that are necessary to obtain the computed allocation as an equilibrium allocation. Third, if one uses global methods to approximate a recursive equilibrium (such as projection methods, see Judd et al. (2003) for an overview), it is important to find simple bounds on the endogenous state space. These bounds are endogenous if one uses financial wealth as the endogenous state. In contrast, for the case of recursive weights, as policies are homogeneous of degree zero in recursive weights, we can take a unit simplex as the endogenous state space. Finally, if one uses local methods, then portfolio choices at the deterministic steady state are not determined; using recursive weights as a state circumvents this problem since a deterministic steady state in weights is well defined and standard perturbation methods can be used to find local approximations to the recursive equilibrium.

When it comes to theory, it is well known that recursive equilibrium might not exist in stochastic models with overlapping generations if one uses beginning-of-period asset holdings as the endogenous state variable (see Kubler and Polemarchakis (2004)). Sufficient conditions for (generic) existence have been developed for the case without borrowing constraints by Citanna and Siconolfi (2010) but they are often not applicable to models used in applications. We examine the existence of equilibria that are Markovian in the (recursive) Negishi weights. While we do not know of counterexamples to existence in our framework, it seems likely that they can be constructed. However, it is known that in our unconstrained model with complete markets, equilibrium is unique if all agents' utility functions satisfy the gross substitute property (see Kehoe et al. (1991)). We show that this property also guarantees the uniqueness of continuation equilibrium and hence the existence of a recursive equilibrium. This is still the case if agents are borrowing constrained (Gottardi and Kubler (2013) consider a related case with infinitely lived agents).

We also consider a version of the model with an atomless and purely transitory shock to agents' discounting and an atomless shock to endowments and/or dividends which does not depend directly on last period's shocks. Following recent advances in stochastic games by Duggan (2012), we give general conditions for the existence of recursive equilibrium in this version of our model. Nowak and Raghavan (1992) prove the existence of correlated Markov equilibria in a class of stochastic games with continuous states. Their method makes two key assumptions. First, they assume that the probability distribution of next period's state varies continuously with current actions; following the literature, we refer to this property as 'norm-continuous transition'. In our setup, this property cannot simply be assumed, yet needs to be derived from economic fundamentals, in particular from the presence of shock to agents' discounting. Second, they introduce a public randomization device (sunspots). Duggan (2012) shows that one can dispense with sunspots if some component of the exogenous state is subject to a shock that is atomlessly distributed and does not depend on the previous state directly (but perhaps indirectly through the other component of the exogenous shock). Assuming that there is a shock to endowments and/or dividends that satisfies these properties we can modify Duggan's proof to fit our framework and prove that recursive equilibria always exist if borrowing against future labor income is not possible. To the best of our knowledge this is the first general existence result for stochastic models with overlapping generations and borrowing constraints. Citanna and Siconolfi (2010, 2012) prove the generic existence of recursive equilibria for complete as well as incomplete markets, yet they need to assume a very large number of heterogeneous agents per generation and their proof cannot be extended to models with borrowing constraints.

In this paper we focus on pure exchange economies. The introduction of a neoclassical production sector is straightforward – however our existence result that relies on a gross-substitute property does not carry over. The existence of recursive equilibria in production economies is an open question for future research.

The rest of the paper is organized as follows: In Section 2 we introduce the model. We start our analysis, in Section 3, with a simple example in which markets are complete and there are finitely many shocks. In Section 4 we give a general definition of recursive equilibrium. In Section 5 we prove the existence of recursive equilibrium under a gross-substitutes assumption on preferences. In Section 6 we prove existence under assumptions that make the state transition norm-continuous.

2 Model

Time is indexed by $t \in \mathbb{N}_0$. Exogenous shocks $z_t \in \mathbf{Z}$ realize in a complete, separable metric space $\mathbf{Z}$, and follow a first-order Markov process with transition probability $\mathbb{P}(.|z)$ defined on the Borel σ-algebra $\mathcal{Z}$ on $\mathbf{Z}$, that is $\mathbb{P} : \mathbf{Z} \times \mathcal{Z} \to [0, 1]$. By a standard argument one can construct a filtration $(\mathcal{F}_t)$ so that (z_t) is an $\mathcal{F}_t$-adapted stochastic process. A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, \ldots, z_t)$ and is also called a date event. Whenever convenient we simply use t instead of z^t. To indicate that z^τ is a successor of z^t or that $z^\tau = z^t$, we write $z^\tau \succeq z^t$. We write (x_t) to denote an $\mathcal{F}_t$-adapted stochastic process.

We consider an exchange economy with overlapping generations. At each date event H agents enter the economy and live for A periods. An agent is identified by the date event of his birth, z^t, and his type, $h \in \mathbf{H} = \{1, \ldots, H\}$. Agent (z^t, h) consumes and receives endowments at all date events $z^{t+a-1} \succeq z^t$, where $a \in \{1, \ldots, A\}$ denotes the age of the agent. At a given date event z^t we can uniquely identify agents who consume at that date event by their age and type, (a, h). We denote the set of all these agents by $\mathbf{A} = \{(a, h) : 1 \leq a \leq A, h \in \mathbf{H}\}$ and the set of all agents except for generation i by $\mathbf{A}_{-i} = \mathbf{A} \setminus \{(a, h) : a = i, h \in \mathbf{H}\}$. We will use (a, h) and (z^t, h) interchangeably to refer to a specific agent.

At each date event there are L perishable commodities, $l \in \{1, ..., L\}$, available for consumption. We denote individual endowments by $\omega_{a,h}(z^t) \in \mathbb{R}^L_{++}$ and assume that they are positive, time-invariant, and measurable functions of the current shock alone. Endowments of agent (a, h) can be decomposed into two components, both time-invariant and measurable functions of the shock,[1] that is

$$\omega_{a,h}(z^t) = e_{a,h}(z_t) + f_{a,h}(z_t) \text{ for all } z^t.$$

We assume that all endowments are bounded, and we call the two components f-endowments and e-endowments respectively. The f-endowments are tangible resources that can be pledged to finance consumption and asset purchases at a time before they are received. In contrast, the e-endowments are non-pledgeable. This formulation generalizes to two widely used modelling assumptions. If $e_{a,h}(z) = 0$ for all (a, h) and all z, the model reduces to the standard model without market imperfections. If $f_{a,h}(z) = 0$ for all (a, h) and all z, then agents cannot borrow against future endowments and all borrowing must be collateralized by the tree (see Chien and Lustig (2010) or Gottardi and Kubler (2013)).

We take the consumption space to be the space of adapted and essentially bounded processes. Each agent has a time-separable utility function

$$U_{z^t,h}(x) = u_{1,h}(x_t) + \mathbb{E}_t \left[\sum_{a=1}^{A-1} (\Pi_{j=1}^{a} \delta_{j+1,h}(z_{t+j})) u_{a+1,h}(x_{t+a}) \right],$$

where $x_{t+a} \in \mathbb{R}^L_+$ denotes (stochastic) consumption of agent (z^t, h) at date $t + a$, and x denotes consumption over the lifecycle

$$x = \{x(z^{t+a})\}_{0 \leq a \leq A-1, z^{t+a} \succeq z^t}.$$

The possibly stochastic discount factors are assumed to be measurable functions bounded below by $\underline{\delta} > 0$ and above by $\overline{\delta} < \infty$, $\delta_{a,h} : \mathbf{Z} \to [\underline{\delta}, \overline{\delta}]$, and might vary with age and type.[2] The Bernoulli-functions $u_{a,h} : \mathbb{R}^L_{++} \to \mathbb{R}$ are assumed to be C^2 on $\mathbb{R}^L_{++}$, strictly increasing, strictly concave, and satisfy an Inada condition: for all $x \in \mathbb{R}^L_+ \setminus \mathbb{R}^L_{++}$ along any sequence $x^n \to x$, $\|D_x u_{a,h}(x^n)\| \to \infty$.

[1]Note that this does not imply that endowments are first-order Markov as the shock is, they may well be higher-order Markov.

[2]For an application with stochastic discount factors, see Krusell and Smith (1997) who use this modelling device to match the wealth distribution.

There is a Lucas tree in unit net supply paying dividends $d(z^t) \in \mathbb{R}^L_+$, $d(z^t) > 0$. Dividends are a function of the shock alone, so $d(z^t) = d(z_t)$ for some measurable function $d : \mathbf{Z} \to \mathbb{R}^L_+$ that is bounded above. At time $t = 0$, in addition to the H new agents (z^0, h), $h \in \mathbf{H}$, there are individuals of each age $a = 2, \ldots, A$ and each type $h = 1, \ldots, H$ present in the economy. We denote these individuals by (z^{1-a}, h) for $h = 1, \ldots, H$ and $a = 2, \ldots, A$. They have initial tree holdings, $\theta_{z^{1-a},h}(z^{-1})$, summing up to one:

$$\sum_{a=2}^{A} \theta_{z^{1-a},h}(z^{-1}) = 1.$$

These holdings determine the 'initial condition' of the economy. The aggregate endowment in the economy is the sum of dividends and endowments $\bar{\omega}(z^t) = \bar{\omega}(z_t) = d(z_t) + \sum_{(a,h)\in\mathbf{A}} \omega_{a,h}(z_t)$.

We define an Arrow–Debreu equilibrium to be a process of $\mathcal{F}_t$-adapted (probability adjusted) prices and choices,

$$\left(p_t, \left(x_{(a,h),t}\right)_{(a,h)\in\mathbf{A}}\right)_{t=0}^{\infty}$$

such that markets clear and agents optimize, that is (A) and (B) hold.

(A) Market clearing equations:

$$\sum_{(a,h)\in\mathbf{A}} x_{a,h}(z^t) = \bar{\omega}(z_t), \quad \text{for all } z^t.$$

(B) For each z^t and $h = 1, \ldots, H$, individual (z^t, h) maximizes utility:

$$x_{z^t,h} \in \arg\max_{x\geq 0} U_{z^t,h}(x) \ \text{ s.t. the constraints (i) and (ii).}$$

(i) Budget constraint:

$$\mathbb{E}_t \left[\sum_{a=0}^{A-1} p(z^{t+a}) \cdot x(z^{t+a})\right] \leq \mathbb{E}_t \left[\sum_{a=0}^{A-1} p(z^{t+a}) \cdot \omega_{a+1,h}(z_{t+a})\right] < \infty.$$

(ii) Borrowing constraint at each age $j = 1, ..., A-1$:

$$\mathbb{E}_{t+j} \left[\sum_{a=j}^{A-1} p(z^{t+a}) \cdot (x(z^{t+a}) - e_{a+1,h}(z_{t+a}))\right] \geq 0.$$

The utility maximization problems for the agents who are initially alive at $t = 0$ (namely (z^k, h) with $k \in \{-(A-1), \ldots, -1\}, h \in \mathbf{H}$) are analogous to the optimization problems for the other agents ((z^t, h) with $t \geq 0, h \in \mathbf{H}$), except that they have a shorter planning horizon and (i) now includes the value of their claims to the infinite dividend stream of the tree. An agent 'born' at time $k < 0$ still consumes until period $A - 1 + k$ and faces the budget constraint

$$\mathbb{E}_0 \left[\sum_{t=0}^{A-1+k} p(z^t) \cdot x(z^t)\right] \leq \mathbb{E}_0 \left[\sum_{t=0}^{A-1+k} p(z^t) \cdot \omega_{t-k,h}(z_t) + \sum_{t=0}^{+\infty} p(z^t) \cdot d(z_t)\theta_{z^k,h}(z^{-1})\right] < \infty.$$

Constraint (ii) is non-standard and reflects the inability of agents to use their future e-endowment for current consumption or investment. Chien and Lustig (2010) introduce this form of limited enforceability, show equivalence between this definition and a definition of sequential equilibrium with collateral constraints on short sales of Arrow securities, and discuss its asset pricing implications.

3 A simple example

To motivate the following analysis it is useful to first discuss the simplest version of the model in some detail. For this section, we assume that $\mathbf{Z} = \{1, \ldots, Z\}$ is a finite set and that all endowments can be pledged, that is $\omega_{a,h}(z) = f_{a,h}(z)$ for all (a, h) and all z. We use $\pi(z'|z)$ to denote $\mathbb{P}(\{z'\}|z)$, $\pi(z^t)$ to denote the probability of history z^t, Σ to denote the event tree, and $\sigma = z^t$ for a typical element of Σ.

Demange (2002) shows that in the presence of a Lucas tree, Arrow–Debreu equilibrium in this model is always Pareto-efficient. As Kehoe et al. (1992) point out, the equilibrium allocation can then be obtained as the solution to a planner's problem. In fact, the presence of a Lucas tree ensures that there must exist summable Pareto weights $\{\eta_{z^t,h}\}_{z^t\in\Sigma,h\in\mathbf{H}}$ such that Arrow–Debreu equilibrium allocations satisfy

$$(x_{z^t,h})_{z^t\in\Sigma,h\in\mathbf{H}} = \arg\max_{x\geq 0} \sum_{z^t\in\Sigma,h\in\mathbf{H}} \pi(z^t)\eta_{z^t,h}U_{z^t,h}(x_{z^t,h}) \quad \text{s.t.} \sum_{z^t\in\Sigma,h\in\mathbf{H}} x_{z^t,h}(\sigma) \leq \bar{\omega}(\sigma) \text{ for all } \sigma \in \Sigma.$$

Since we assume time-separable expected utility, we can characterize equilibrium also by using *instantaneous Negishi weights*, $\lambda(z^t) = (\lambda_{a,h}(z^t))_{(a,h)\in\mathbf{A}}$, which we define by

$$\lambda_{1,h}(z^t) = \eta_{z^t,h}, \quad \lambda_{a,h}(z^t) = \lambda_{a-1,h}(z^{t-1})\delta_{a,h}(z_t), \ a = 2, ..., A.$$

Individuals' consumption is then given as a function $X : \mathbf{Z} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AHL}_{+}$ of the shock and the instantaneous weights with

$$X(z, \lambda) = \arg\max_{x\in\mathbb{R}^{AHL}_{+}} \sum_{(a,h)\in\mathbf{A}} \lambda_{a,h}u_{a,h}(x_{a,h}) \text{ s.t.} \sum_{(a,h)\in\mathbf{A}} x_{a,h} \leq \bar{\omega}(z). \tag{1}$$

Note that for a given z, $X(z, .)$ is a continuous function. For a process of instantaneous Negishi weights $(\lambda(z^t))_{z^t\in\Sigma}$, $\lambda(z^t) \in \mathbb{R}^{AH}_{++}$ for all z^t, we define for each node z^t, $x_{a,h}(z^t) := X_{a,h}(z_t, \lambda(z^t))$. Then a sequence of instantaneous weights

$$\left((\lambda_{a,h}(z^t))_{(a,h)\in\mathbf{A}}\right)_{z^t\in\Sigma}$$

characterizes an Arrow–Debreu equilibrium if the following two conditions hold.

- Evolution of instantaneous weights:
  For all $h \in \mathbf{H}$, all $a = 2, ..., A$, and all $z^t \in \Sigma$, it holds that $\lambda_{a,h}(z^t) = \delta_{a,h}(z_t)\lambda_{a-1,h}(z^{t-1})$.

- Budget constraints:

  Defining the budget of agent (a,h) at node z^t for all $h \in \mathbf{H}$ recursively by

  $$w_{A,h}(z^t) := D_x u_{A,h}(x_{A,h}(z^t)) \cdot \left(x_{A,h}(z^t) - \omega_{A,h}(z^t)\right), \text{ and for } a = 1, ..., A-1 \text{ by}$$

  $$w_{a,h}(z^t) := D_x u_{a,h}(x_{a,h}(z^t)) \cdot \left(x_{a,h}(z^t) - \omega_{a,h}(z^t)\right) + \sum_{z^{t+1} \succeq z^t} \pi(z_{t+1}|z_t)\delta_{a+1,h}(z_{t+1}) w_{a+1,h}(z^{t+1}),$$

  it holds for all $h \in \mathbf{H}$ that $w_{1,h}(z^t) = 0$.

Note that the budgets $w_{a,h}(z^t)$ are functions of $\lambda(z^t)$ as they are functions of $x_{a,h}(z^t) = X_{a,h}(z_t, \lambda(z^t))$. It is easy to verify that for a sequence of instantaneous Negishi weights that satisfies the above conditions, there exist initial conditions and an Arrow–Debreu equilibrium,

$$\left(\bar{p}(z^t), \left(\bar{x}_{a,h}(z^t)\right)_{(a,h) \in \mathbf{A}}\right)_{z^t \in \Sigma},$$

with $\bar{x}_{a,h}(z^t) = X_{a,h}(z_t, \lambda(z^t))$.

Equilibrium prices are unique up to a normalization. Since by the definition of the Negishi problem in (1), $\lambda_{a,h}(z^t) D_x u_{a,h}(x_{a,h}(z^t))$ are identical across all agents $(a,h) \in \mathbf{A}$, a convenient way to express Arrow–Debreu prices as a function of Negishi weights is to define $p(z^t) = \lambda_{1,1}(z^t) D_x u_{1,1}(x_{1,1}(z^t))$.

3.1 Recursive equilibria

Using Negishi's approach to compute equilibria is useful only if the instantaneous Negishi weights follow a Markov process. In this case, one can numerically approximate their transition and the consumption allocation can be easily computed from (1). In our definition of recursive equilibrium below, we require slightly more in that we want the instantaneous weights of the agents that enter the economy to depend only on the weights of the currently alive. The weight of the old in the last period should play no role in this. This is in the spirit of recursive methods where the current endogenous state typically depends only on current variables determined in the previous period. We therefore use discounted previous-period instantaneous Negishi weights of the non-newborn agents as the endogenous state. We refer to these as *recursive weights* and denote them by $\tilde{\lambda} = (\tilde{\lambda}_{a,h})_{(a,h) \in \mathbf{A}_{-1}}$. Note that, for the complete-markets case that we are currently considering, the instantaneous weight of a (non-newborn) agent is just equal to his recursive weight, which will no longer be the case when we consider borrowing-constraint economies where the instantaneous weight is larger than the recursive weight whenever the borrowing constraint binds.

As policies are homogeneous of degree zero in recursive Negishi weights, we can normalize the state to lie in the $(A-1)H-1$ dimensional unit simplex, $\Delta^{(A-1)H-1} = \{x \in \mathbb{R}_+^{(A-1)H} : \sum_{i=1}^{(A-1)H} x_i = 1\}$. Since agents are finitely lived, our assumptions on the utility function imply in fact that we can focus attention on a closed set $\mathbf{\Lambda} \subset \Delta^{(A-1)H-1}$ with $\mathbf{\Lambda} \subset \mathbb{R}_{++}^{(A-1)H}$. The state space is then $\mathbf{S} = \mathbf{Z} \times \mathbf{\Lambda}$ with a typical element $s = (z, \tilde{\lambda})$. Given $\tilde{\lambda}$, we denote by $\gamma_{1,h} : \mathbf{S} \to \mathbb{R}_+$ the function mapping the state into the instantaneous Negishi weights of the newborn of type h. Note that in addition to the endogenous state $\tilde{\lambda}$, the instantaneous Negishi weights of the newborn are needed to determine consumption, $X : \mathbf{Z} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AHL}_{+}$, defined by (1). To define recursive equilibrium, we first need to write the budget set recursively. We denote by $W_{a,h}(s)$ the value of an agent's current and future consumption in excess of his individual endowments (in a sequential formulation, this will simply be his financial wealth). We define a recursive equilibrium to consist of functions $\gamma_1 : \mathbf{S} \to \mathbb{R}^{H}_{+}$ and $W : \mathbf{S} \to \mathbb{R}^{AH}$ such that for all $s \in \mathbf{Z} \times \mathbf{\Lambda}$, and for all $h \in \mathbf{H}$,

$$
\begin{aligned}
W_{A,h}(s) &= D_x u_{A,h}(X_{A,h}(z,\lambda)) \cdot (X_{A,h}(z,\lambda) - \omega_{A,h}(z)) \\
W_{a,h}(s) &= D_x u_{a,h}(X_{a,h}(z,\lambda)) \cdot (X_{a,h}(z,\lambda) - \omega_{a,h}(z)) + \sum_{z'} \pi(z'|z)\delta_{a+1,h}(z')W_{a+1,h}(s'),\ a = 1, ..., A-1 \\
W_{1,h}(s) &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda &= \left((\gamma_{1h}(z,\tilde{\lambda}))_{h\in\mathbf{H}}, \tilde{\lambda}\right) \\
s' &= (z', \tilde{\lambda}') \text{ with } \tilde{\lambda}'_{\bar{a},\bar{h}} = \frac{\delta_{\bar{a},\bar{h}}(z')\lambda_{\bar{a}-1,\bar{h}}}{\sum_{(a,h)\in\mathbf{A}_{-1}} \delta_{a,h}(z')\lambda_{a-1,h}} \text{ for all } (\bar{a},\bar{h}) \in \mathbf{A}_{-1}, \text{ and with } \tilde{\lambda}' \in \mathbf{\Lambda}.
\end{aligned}
$$

It is easy to verify that a recursive equilibrium, if it exists, describes an Arrow–Debreu equilibrium. We discuss the existence of recursive equilibrium in Section 4. For now, we take its existence for granted and point out the computational advantages of using recursive Negishi weights as the endogenous state variable.

3.2 Computation

We describe and discuss a simple time iteration collocation method to numerically approximate recursive equilibria. Time iteration is one of several standard approaches to solving dynamic models (see e.g., Section 7.2. of Judd et al. (2003) or Krueger and Kubler (2004)). Obviously there are several other approaches which have advantages and disadvantages compared to time iteration (see in particular Rios-Rull (1996)). However, we choose to discuss this algorithm because it allows for a simple comparison of our approach to the conventional approach of doing time iteration using the natural state space. It also serves as a basis for computing large-scale models in practice. We conjecture that our Negishi approach can also be usefully employed with other computational methods such as the linearization technique in Rios-Rull (1996) or perturbation methods.

We take as given that the functions $X(z, \lambda)$ can be approximated with high accuracy and negligible computational cost. For standard calibrations that assume identical homothetic utility this function is linear after a change of variable. For the case of one commodity there are several other classes of preferences for which closed-form solutions are known.

We assume that the unknown functions $W(z, \tilde{\lambda})$ and $\gamma_{1,h}(z, \tilde{\lambda})$ can be well approximated by some

$\hat{W}(z, \tilde{\lambda})$ and $\hat{\gamma}(z, \tilde{\lambda})$ that are uniquely determined by the requirement that $\hat{W}_{a,h}(z, \tilde{\lambda}^i) = W_{a,h}(z, \tilde{\lambda}^i)$ and $\hat{\gamma}_h(z, \tilde{\lambda}^i) = \gamma_h(z, \tilde{\lambda}^i)$ for all $z \in Z$ and some finite number G of so-called 'collocation points' $\tilde{\lambda}^i \in \mathbf{G} \subset \Delta^{(A-1)H-1}$, $i = 1, ..., G$. Examples of interpolation schemes used for collocation include splines as in Judd et al. (2003), Smolyak polynomials as in Krueger and Kubler (2004), Delaunay interpolation as in Brumm and Grill (2014), and piecewise multi-linear (hierarchical) basis functions on (adaptive) sparse grids as in Brumm and Scheidegger (2013). The main steps of the algorithm are as follows:

1. Set $n = 0$ and start with an initial guess $\hat{W}^0 : \mathbf{Z} \times \mathbf{\Lambda} \to \mathbb{R}^{AH}$.

2. Given $\hat{W}^n$, for each $z \in \mathbf{Z}$ and each $\tilde{\lambda}^i \in \mathbf{G}$, compute $\hat{\gamma}^{n+1}(z, \tilde{\lambda}^i)$ as a solution to the system of equations

$$D_x u_{1,h}(X_{1,h}(z, \lambda^i)) \cdot (X_{1,h}(z, \lambda^i) - \omega_{1,h}) + \sum_{z'} \pi(z'|z)\delta_{2,h}(z')\hat{W}^n_{2,h}(z', \tilde{\lambda}'(z')) = 0, \ \ h \in \mathbf{H}, \quad (2)$$

where

$$\lambda^i = (\hat{\gamma}^{n+1}(z, \tilde{\lambda}^i), \tilde{\lambda}^i) \text{ and } \tilde{\lambda}'_{\bar{a},\bar{h}}(z') = \frac{\delta_{\bar{a},\bar{h}}(z')\lambda^i_{\bar{a}-1,\bar{h}}}{\sum_{(a,h)\in\mathbf{A}_{-1}} \delta_{a,h}(z')\lambda^i_{a-1,h}} \text{ for all } (\bar{a}, \bar{h}) \in \mathbf{A}_{-1}.$$

3. For each $z \in \mathbf{Z}$ and each $\tilde{\lambda}^i \in \mathbf{G}$, let λ^i and $\tilde{\lambda}'(z')$ be given as before and compute for all $(a, h) \in \mathbf{A}$

$$\hat{W}^{n+1}_{a,h}(z, \tilde{\lambda}^i) = D_x u_{a,h}(X_{a,h}(z, \lambda^i)) \cdot (X_{a,h}(z, \lambda^i) - \omega_{a,h}) + \sum_{z'} \pi(z'|z)\delta_{a+1,h}(z')\hat{W}^n_{a+1,h}(z', \tilde{\lambda}'(z')),$$

where $\hat{W}^{n+1}_{A+1,h}(z, \tilde{\lambda}^i) := 0$.

4. For each $z \in \mathbf{Z}$, interpolate $\{\hat{W}^{n+1}(z, \tilde{\lambda}^i), i = 1, ..., G\}$ to obtain approximating functions $\hat{W}^{n+1}(z, .)$.

5. Check some error criterion. If the error criterion is not met, increase n by 1 and go to 2.

6. Set $\hat{W}^* = \hat{W}^{n+1}$ and interpolate $\hat{\gamma}^{n+1}(z, \tilde{\lambda}^i)$ to obtain $\hat{\gamma}^*(z, .)$.

In general, the system of equations (2) might have no solutions, or it might have several solutions. Furthermore, there is no guarantee that $\hat{W}^n$ converges as n tends to infinity. This is obvious as there is generally no guarantee that a (recursive) equilibrium exists. Feng et al. (2013) develop a method which can be used to compute 'generalized Markov equilibria', which in our setup would be equilibria that are recursive in $(\tilde{\lambda}, W)$. However, for reasonable values of A the method is not applicable as it suffers from a severe curse of dimensionality.

In this simple framework with complete markets, it is clear that using recursive Negishi weights as the endogenous state variable has important advantages over the 'standard' approach that uses beginning-of-period financial wealth.[3] Most strikingly, the computational complexity barely increases with the number of goods, L. Only the computation of $X(z,\lambda)$ and of $D_x u_{a,h}(X(z,\lambda))$ depends on L. This is in stark contrast to the case of financial wealth as endogenous state variable where one needs to solve for spot-prices and allocations simultaneously with portfolios and asset-prices.

Even for the case of a single commodity the Negishi approach has three important advantages over conventional methods: First, as policies are homogeneous of degree zero in recursive Negishi weights, policy functions may be defined over the $(A-1)H-1$ dimensional unit simplex (for given states today and tomorrow). Thus the admissible set is simple and can easily be worked with, while it can be arbitrarily complicated in the case of the natural state space. Since agents do not face borrowing constraints, young agents typically borrow substantial amounts and one thus has to determine the 'natural' borrowing limits as one step of the computations. For models with large H and/or A this can result in substantial difficulties. Second, along the time iteration the only costly computation consists in solving for $\hat{\gamma}(z,\tilde{\lambda}^i)$ in Step 2 above. For each z and $\tilde{\lambda}^i$, this is a non-linear system of H equations in as many unknowns. In contrast, for the case of financial wealth, one needs to solve all agents' first order conditions plus market clearing conditions simultaneously to obtain optimal choices and prices. This results in $(A-1)HZ$ equations for each z and $\tilde{\lambda}^i$ (if market clearing conditions are used to express one agent's portfolio in terms of all others'). Even for moderate A and Z this can be an order of magnitude larger, thus an enormous efficiency gain can be realized if recursive Negishi weights are used. Note also that the dynamics of the economy are fully captured by the HZ functions $(\gamma_h(z,.))_{z\in\mathbf{Z},h\in\mathbf{H}}$. If financial wealth is used instead, one needs to keep track of $(A-1)HZ^2$ functions, for each current shock z mapping financial wealth of all generations but the oldest into their financial wealth at all successor nodes. Thus, the Negishi approach reduces both the number of equations that have to be solved simultaneously and the number of functions that are needed to characterize equilibrium dynamics by a factor of $(A-1)Z$. Third, error analysis is trivial to conduct if we use recursive weights. As mentioned above, the error in computing $X(z,\lambda)$ can typically be taken to be negligible. Given a transition $\hat{\gamma}$, the errors in the computation of $\hat{W}$ are pure function-approximation errors and there are reliable methods to bound them above. As explained for example in Kubler and Schmedders (2005) it is generally impossible to find bounds on how close a computed approximation is to an exact equilibrium. In the current context, it is impossible to determine how close the computed evolution of λ is to the exact equilibrium evolution. However, given approximations $\hat{W}$ and $\hat{\gamma}$ for the unknown policy functions, the only relevant error is

$$MAXERR = \sup_{h\in\mathbf{H},z\in\mathbf{Z},\tilde{\lambda}\in\mathbb{R}_{++}^{(A-1)H}} \|\hat{W}_{1,h}(z',\hat{\gamma}(z,\tilde{\lambda}))\cdot\left(\frac{\partial u_{a,h}(X_{a,h}(s,\lambda))}{\partial x_1}\right)^{-1}\|.$$

[3]Note that in OLG models the assumption of complete markets does not simplify the analysis as much as in models with infinitely lived agents, because agents cannot trade before birth.

This error can be interpreted as the maximal transfer necessary to turn the computed allocation into an equilibrium allocation. That is, while we cannot guarantee in general that the computed allocation is close to an exact equilibrium allocation, it is always close to an equilibrium allocation of an economy with transfers. The size of the transfers is bounded by $MAXERR$. Kubler and Schmedders (2005) suggest a similar interpretation for the case of the natural state. However, in their method one needs to transform the error in the computation into an error that has an economic interpretation. Using recursive Negishi weights as the state variable has the advantage that the error in the computation translates directly to an interpretable approximation error.

4 Negishi's approach in borrowing-constrained economies

We now turn to the general model where we allow for positive e-endowments and for a more general shock structure. We first characterize Arrow–Debreu equilibria in terms of instantaneous Negishi weights and then use this characterization to define recursive equilibrium.

Note that individuals' consumption as a function of instantaneous Negishi weights and the shock, $X(z, \lambda)$ as defined in (1), is continuous in λ for a given z, and measurable in (z, λ) by the Measurable Maximum Theorem (see Theorem 18.19 in Aliprantis and Border (2006)). Given any $\mathcal{F}_t$-adapted process of instantaneous Negishi weights (λ_t) with $\lambda(z^t) \in \mathbb{R}^{AH}_{++}$ for all z^t, we write for each z^τ, $x_{a,h}(z^\tau) = x_{a,h}(z^\tau; (\lambda_t)) = X_{a,h}(z_\tau, \lambda(z^\tau))$. By definition of $X_{a,h}(z, .)$ we have that $\lambda_{a,h}(z^\tau) D_x u_{a,h}(x_{a,h}(z^\tau))$ is identical across all agents $(a, h) \in \mathbf{A}$. For each $h \in \mathcal{H}$, we define

$$w_{A,h}(z^\tau, (\lambda_t)) = \lambda_{A,h}(z^\tau) D_x u_{A,h}(x_{A,h}(z^\tau)) \cdot (x_{A,h}(z^\tau) - \omega_{A,h}(z_\tau))$$

and for all $a = 1, ..., A-1$

$$w_{a,h}(z^\tau; (\lambda_t)) = \lambda_{a,h}(z^\tau) D_x u_{a,h}(x_{a,h}(z^\tau)) \cdot (x_{a,h}(z^\tau) - \omega_{a,h}(z_\tau)) + \int w_{a+1,h}(z^{\tau+1}, (\lambda_t)) d\mathbb{P}(z_{\tau+1}|z_\tau).$$

Using e-endowments instead of total endowments, we define for each $h \in \mathcal{H}$

$$v_{A,h}(z^\tau; (\lambda_t)) = D_x u_{a,h}(x_{a,h}(z^\tau)) \cdot (x_{a,h}(z^\tau) - e_{a,h}(z_\tau)),$$

and for $a = 1, ..., A-1$,

$$v_{a,h}(z^\tau; (\lambda_t)) = D_x u_{a,h}(x_{a,h}(z^\tau)) \cdot (x_{a,h}(z^\tau) - e_{a,h}(z_\tau)) + \int \delta_{a+1,h}(z_{\tau+1}) v_{a+1,h}\left(z^{\tau+1}, (\lambda_t)\right) d\mathbb{P}(z_{\tau+1}|z_\tau).$$

Given an Arrow–Debreu equilibrium with consumption allocation $(x_{(a,h),t})$ it is clear that there exists a unique $\lambda(z^0) \in \Delta^{AH-1}$ and for each date event z^t, $t > 0$ a unique $\lambda(z^t) \in \mathbb{R}^{(AH)}_{++}$ such that: for all $(a, h) \in \mathbf{A}_{-1}$, we have $\lambda_{a,h}(z^t) \geq \delta_{a,h}(z_t) \lambda_{a-1,h}(z^{t-1})$, where the inequality holds as an equality for at least one (a, h), and for all z^t

$$x(z^t) \in \arg\max \sum_{(a,h)\in\mathbf{A}} \lambda_{a,h}(z^t) u_{a,h}(x_{a,h}) \text{ s.t. } \sum_{(a,h)\in\mathbf{A}} x_{a,h} - \bar{\omega}(z_t) \leq 0.$$

Note that we can take $p(z^t) = \lambda_{1,1}(z^t) D_x u_{1,1}(x_{1,1}(z^t))$ — at least one agent must be unconstrained between node z^{t-1} and node z^t and thus prices must be equal to this agent's marginal rates of substitution. By the agents' budget constraints, we must have that the equilibrium process of weights, (λ_t), satisfies

$$w_{1,1}\left(z^\tau;(\lambda_t)\right) = 0 \text{ for all } z^\tau. \tag{3}$$

Furthermore note that for any $z^{\tau+1}$ and any $(a,h) \in \mathbf{A}_{-A}$

$$\frac{\delta_{a+1,h}(z_{\tau+1}) D_x u_{a,h}(x_{a+1,h}(z^{\tau+1}))}{D_{x_1} u'_{a,h}(x_{a,h}(z^\tau))} \neq \frac{p(z^{\tau+1})}{p(z^\tau)} \text{ only if } v_{a+1,h}(z^{\tau+1};(\lambda_t)) = 0.$$

Therefore the borrowing constraint implies that for all $(a,h) \in \mathbf{A}_{-1}$ and all z^τ

$$v_{a,h}\left(z^\tau;(\lambda_t)\right) \geq 0, \quad (\lambda_{a,h}(z^\tau) - \delta_{a,h}(z_\tau)\lambda_{a-1,h}(z^{\tau-1})) v_{a,h}\left(z^\tau;(\lambda_t)\right) = 0. \tag{4}$$

What is important for our analysis is that the converse is also true, namely that any sequence of instantaneous weights that satisfy (3) and (4) describes an Arrow–Debreu equilibrium. If (z_t, λ_t) follows a Markov process it suffices to numerically approximate the transition for λ and then back out the equilibrium allocations and prices from the transition. The following lemma formalizes the claim.

Lemma 1 *Given a process of instantaneous Negishi weights, (λ_t), that satisfies for all z^t, $t > 0$, and all $(a,h) \in \mathbf{A}_{-1}$,*

$$\lambda_{a,h}(z^t) \geq \delta_{a,h}(z_t)\lambda_{a-1,h}(z^{t-1}),$$

suppose that (3) and (4) hold. Then there exist initial conditions, $\theta_{a,h}(z^{-1})$, and an Arrow–Debreu equilibrium, (x_t, p_t), such that for all z^t the consumption allocation and prices are given by

$$x(z^t) = X(z_t, \lambda(z^t)), \quad p(z^t) = \lambda_{1,1}(z^t) D_x u_{1,1}(x_{1,1}(z^t)).$$

Proof. By construction, $x(z^\tau, (\lambda_t))$ satisfies market-clearing and all agents' budget constraints. We need to prove that each agent maximizes utility subject to the budget and borrowing constraints. For each agent (z^t, h) we define the Lagrangian associated with his optimization problem as

$$\begin{aligned} \mathcal{L}(x,\ell) = \quad - \quad U_{z^t,h}(x) \quad + \quad \ell_0 \ \mathbb{E}_t \left[\sum_{a=0}^{A-1} p(z^{t+a}) \cdot (x(z^{t+a}) - \omega_{a+1,h}(z_{t+a})) \right] \\ - \quad \mathbb{E}_t \left[\sum_{j=1}^{A-1} \ell_j(z^{t+j}) \sum_{a=j}^{<} A - 1 p(z^{t+a}) \cdot (x(z^{t+a}) - e_{a+1,h}(z_{t+a})) \right]. \end{aligned}$$

Therefore, $\partial \mathcal{L}(x,\ell)/\partial x(z^{t+a})$ is given by

$$\begin{aligned} & -\mathbb{E}_t \left[(\Pi_{j=1}^a \delta_{j+1,h}(z_{t+j})) D_x u_{a+1,h}(x_{a+1,h}(z^{t+a}) \right] + \ell_0 \ \mathbb{E}_t \left[p(z^{t+a}) \right] - \mathbb{E}_t \left[\sum_{j=1}^{a} \ell_j(z^{t+j}) p(z^{t+a}) \right] \\ = \quad & \mathbb{E}_t \left[D_x u_{a+1,h}(x_{a+1,h}(z^{t+a}) \left(-(\Pi_{j=1}^a \delta_{j+1,h}(z_{t+j})) + \lambda_{a+1,h}(z^{t+a}) \left(\ell_0 - \sum_{j=1}^{a} \ell_j(z^{t+j}) \right) \right) \right], \end{aligned}$$

where we used that $p(z^t) = \lambda_{1,1}(z^t) D_x u_{1,1}(x_{1,1}(z^t)) = \lambda_{a,h}(z^t) D_x u_{a,h}(x_{a,h}(z^t))$ for all (a,h)—the first equality is just our definition of $p(z^t)$ while the second equality follows from $x(z^t) = X(z_t, \lambda(z^t))$. To satisfy $\partial \mathcal{L}(x,\ell)/\partial x(z^{t+a}) = 0$ for all $0 \le a \le A-1$, $z^{t+a} \succeq z^t$ we thus have to define $\ell_0^* = \frac{1}{\lambda_{1,h}(z^t)}$ and for $a = 1, ..., A-1$,

$$\ell_a^*(z^{t+a}) = \left(\frac{1}{\delta_{a+1,h}(z_{t+a}) \lambda_{a,h}(z^{t+a-1})} - \frac{1}{\lambda_{a+1,h}(z^{t+a})} \right) (\Pi_{j=1}^a \delta_{j+1,h}(z_{t+j})).$$

Clearly, (x, ℓ^*) is then a saddle point for $\mathcal{L}$. Therefore x must be an optimal solution to the agent's maximization problem (see e.g., Luenberger (1969) Chapter 8.4). □

4.1 Definition of recursive equilibrium

In a model with borrowing constraints, the recursive weights that we use as the endogenous state are no longer equal to the instantaneous weights of the (non-newborn) agents. They are rather the discounted previous-period instantaneous Negishi weights of the currently alive. To determine the instantaneous weights from the recursive weights, we need to solve functional equations that represent the borrowing constraints. The formal details are as follows.

We fix an endogenous state space $\mathbf{\Lambda} \subset \Delta^{(A-1)H-1}$ and take $\mathbf{S} = \mathbf{Z} \times \mathbf{\Lambda}$. A recursive equilibrium consists of measurable 'value'-functions

$$W : \mathbf{S} \to \mathbb{R}^{AH}, \quad V : \mathbf{S} \to \mathbb{R}_+^{AH},$$

as well as 'policy'-functions

$$\gamma : \mathbf{S} \to \mathbb{R}_+^{AH},$$

such that the following functional equations hold for all $s = (z, \tilde{\lambda}) \in \mathbf{S}$:

$$\begin{aligned} W_{a,h}(s) &= \lambda_{a,h} \left(D_x u_{a,h}(X_{a,h}(z,\lambda)) \cdot (X_{a,h}(z,\lambda) - \omega_{a,h}(z)) + \int \frac{\delta_{a+1,h}(z')}{\tilde{\lambda}'_{a+1,h}} W_{a+1,h}(s') d\mathbb{P}(z'|z) \right) \\ V_{a,h}(s) &= D_x u_{a,h}(X_{a,h}(z,\lambda)) \cdot (X_{a,h}(z,\lambda) - e_{a,h}(z)) + \int \delta_{a+1,h}(z') V_{a+1,h}(s') d\mathbb{P}(z'|z) \end{aligned}$$

for all $(a,h) \in \mathbf{A}$, where $W_{A+1,h}(s) = V_{A+1,h}(s) = 0$ for all s, and

$$\begin{aligned} \lambda &= \left((\gamma_{1h}(z,\tilde{\lambda}))_{h \in \mathbf{H}}, (\tilde{\lambda}_{a,h} + \gamma_{a,h}(z,\tilde{\lambda}))_{(a,h) \in \mathbf{A}_{-1}} \right) \\ s' &= (z', \tilde{\lambda}') \text{ with } \tilde{\lambda}'_{\bar{a},\bar{h}} = \frac{\delta_{\bar{a},\bar{h}}(z') \lambda_{\bar{a}-1,\bar{h}}}{\sum_{(a,h) \in \mathbf{A}_{-1}} \delta_{a,h}(z') \lambda_{a-1,h}} \text{ for all } (\bar{a},\bar{h}) \in \mathbf{A}_{-1}, \text{ and } \tilde{\lambda}' \in \mathbf{\Lambda} \\ W_{1,h}(s) &= 0 \quad \text{for all } h \in \mathbf{H} \\ V_{a,h}(s) &\ge 0, \quad \gamma_{a,h}(s) \cdot V_{a,h}(s) = 0 \quad \text{for all } (a,h) \in \mathbf{A}_{-1}. \end{aligned}$$

Again, given our discussion above, it is easy to see that a recursive equilibrium, if it exists, describes an Arrow–Debreu equilibrium. Note that the definition of W somewhat differs from the definition of $w(z^\tau, (\lambda_t))$ above — since we normalize $\tilde{\lambda}$ to lie in the unit simplex, we need to 'undo' the normalization to obtain a correct price-system.

4.2 Computation

In Section 3.2 we showed for the case of unconstrained borrowing that there are several computational advantages of using instantaneous Negishi weights rather than financial wealth as a state variable. Even in the presence of borrowing constraints, most of these advantages remain. Computational complexity still barely increases in the number of goods, also the state space remains a unit simplex, and the approximation errors have the same straightforward interpretation. However, we now have to solve for the V-functions which involves a non-linear complementarity problem of size $H(A-1)$. Thus, in contrast to the case of unconstrained borrowing, the size of the system now depends on A. Yet in contrast to the case of financial wealth as the state variable, the size of the system still does not depend on S. Moreover, for many realistic calibrations one can expect the borrowing constraint to bind rarely for older generations which effectively makes the size of the system that is to be solved smaller.

4.3 Existence

In general, the existence of recursive equilibria in non-optimal economies is an open problem. For the case of beginning-of-period asset holdings as the endogenous state, Kubler and Polemarchakis (2004) provide simple counter-examples which show that one cannot hope for general existence proofs. Citanna and Siconolfi (2010) have the important insight that with sufficiently many agents per generation recursive equilibrium must exist generically in a model without borrowing constraints. In Citanna and Siconolfi (2012) they show that the same argument is applicable for models with incomplete markets. They prove their result using the natural state space, but it seems likely that a similar approach can be used to prove the generic existence of equilibria with Markovian Negishi weights as long as there are no borrowing constraints (i.e. e-endowments are zero). However, we want to use the fact that Negishi weights are Markov to approximate equilibria using Negishi's method in models where agents live for many periods. In this case, the number of agents needed for Citanna and Siconolfi's result to apply becomes astronomical very quickly (they assume finitely many shocks and require more than Z^A agents, so even with only two exogenous shocks and 60-period-lived agents, one needs more than 10^{18} agents per generation). The use of their approach to tackle models with a continuum of ex ante identical agents within each generation is a subject for further research; for many other models it seems of limited practical relevance. From a theoretical point of view our existence results can be thought of as being complementary to Citanna and Siconolfi's result: Their method assumes few shocks relative to the number of agents, we fix the number of agents but require shocks to be continuous or make strong assumptions on preferences.

In this paper we provide two approaches to prove the existence of recursive equilibria. A first approach is to take as given that Arrow–Debreu equilibria always exist for arbitrary initial instantaneous weights. This can be shown with standard techniques (see, e.g., Kubler and Polemarchakis

(2004) or for a more general shock-structure Aliprantis et al. (1990) or Florenzano et al. (2001)). Recursive equilibria might fail to exist because for a given $(z, \tilde{\lambda})$ there are multiple Arrow–Debreu equilibria and given different histories different ones must be chosen to be consistent with intertemporal optimality. The problem can be solved by making strong enough assumptions on fundamentals that guarantee the uniqueness of continuation equilibria. Following Kehoe et al. (1991) and Dana (1993) it is well known that the assumption of gross substitutes often leads to strong uniqueness results. However, Azariadis and Lambertini (2003) show that, in models with overlapping generations and enforcement constraints, a multiplicity of equilibria can arise even when preferences satisfy a gross-substitutes assumption. It turns out that since in our model borrowing constraints are exogenous the assumption suffices to guarantee uniqueness — we will formalize this in Section 5 below.

A natural second approach tries to use a version of Schauder's fixed point theorem for infinite dimensional spaces to prove the existence of a solution to the functional equation that defines recursive equilibrium. Given functions $W'(z, \tilde{\lambda})$ and $V'(z, \tilde{\lambda})$ for next period, the equilibrium conditions determine functions W and V as well as $\gamma(.)$ for the current period. The problem is that generally the fact that V' and W' are continuous functions in $\tilde{\lambda}$ will not be enough to guarantee that W and V will be continuous functions — the possible multiplicity of solutions makes it unlikely that there exists a continuous selection. However, once one drops continuity of V' and W', one will generally not be able to show that a γ, V, and W exist for this period that satisfy the equilibrium assumptions. A large literature on existence of Markov perfect Nash equilibrium in stochastic games offers a clever solution to this problem. If probability distributions are non-atomic and if the action this period affects the probability distribution over the state next period in a continuous fashion, then the conditional expectation of next period's value function will be continuous in actions today, even if it is discontinuous in the state tomorrow (see e.g., Nowak and Raghavan (1992)). This assumption is referred to as *norm-continuous transition.* In Section 6 we provide conditions under which the endogenous state transition in a recursive equilibrium of our model is norm-continuous. Under these conditions, we prove the existence of a recursive equilibrium by adapting the proof strategy of Duggan (2012) to our setup.

5 Existence with gross substitutes

We show that under a gross-substitutes assumption recursive equilibria always exist. In Pareto optimal exchange economies Arrow–Debreu equilibria are unique, if all goods are pairwise gross substitutes (see Kehoe et al. (1991)). However, it is far from obvious that this is a sufficient condition in the presence of borrowing constraints (see Azariadis and Lambertini (2003)) and overlapping generations. Gottardi and Kubler (2013) consider an exchange economy with infinitely lived agents and a single commodity and their borrowing constraints are equivalent to ours if we set

f-endowments to zero. They show that the assumption of gross substitutes guarantees existence of recursive equilibria (also using recursive weights as a state variable). It turns out that the presence of f-endowments and overlapping generations pose substantial additional difficulties.

To prove existence of recursive equilibria in our model we first need to modify the assumption of gross substitutes for our framework. Dana (1994) shows that with an infinite number of goods, it is more convenient to work with the gross-substitutes property on so-called 'excess utility functions' — $D_x u_{a,h}(X_{a,h}(z,\lambda)) \cdot (X_{a,h}(z,\lambda) - \omega_{a,h}(z))$ in our notation — rather than excess demand functions. For this case, it is useful to define the property as follows (note that the inequality is in the opposite direction compared to the demand case).

DEFINITION 1 *A function $F : \mathbb{R}^m_+ \to \mathbb{R}^m$ satisfies the strict gross-substitutes property if for all $y \in \mathbb{R}^m_+$ and all $x \in \mathbb{R}^m_+ \setminus \{0\}$ with $x_i = 0$ for some $i = 1, ..., m$ it holds that $F_i(y) > F_i(y + x)$. It satisfies the weak gross-substitutes property if the inequality holds weakly.*

We define the set of fundamentals, $\mathbf{F} \subset \mathbb{R}^L_{++}$, to be the set of all possible realization of endowments and dividends, i.e. $\mathbf{F} = \{e_{a,h}(z), f_{a,h}(z), d(z); (a,h) \in \mathbf{A}, z \in \mathbf{Z}\}$. The gross-substitutes assumption can then be formulated as follows.

ASSUMPTION 1 *For all agents (a,h) and all shocks $z \in \mathbf{Z}$, the function $D_x u_{a,h}(X_{a,h}(z,\lambda)) \cdot X_{a,h}(z,\lambda)$ satisfies the weak gross-substitutes property in λ. Moreover, for each $y \in \mathbf{F} \setminus \{0\}$ the function $-D_x u_{a,h}(X_{a,h}(z,\lambda)) \cdot y$ satisfies the strict gross-substitutes property in λ.*

For the case of time-separable utility and a single good per date event, it is easy to see that this assumption is satisfied whenever all agents' relative risk aversion is less than or equal to one — with one commodity one only needs to ensure that $cu'(c)$ is increasing in c, concavity always ensures that $yu'(c)$ is decreasing in c. In our setup with several commodities, however, utility is no longer separable between goods, even if it is time-separable. Assumption 1 can thus become more difficult to verify, yet there are two important special cases for which it is clearly satisfied. First, if utility is separable across all commodities, i.e. for all (a,h) we have $u_{a,h}(x) = \sum_l \hat{u}_{a,h,l}(x_l)$ for some concave functions $\hat{u}_{a,h,l}(.)$, then Assumption 1 obviously reduces to $\frac{\hat{u}''_{a,h,l}(x_l)}{\hat{u}'_{a,h,l}(x_l)} \leq 1$. Second, and more interestingly, if for each (a,h) utility can be written as

$$u_{a,h}(x) = \left(\sum_l x_l^{\zeta^1} \right)^{\zeta^2_{a,h}} \text{ with } \zeta^1, \zeta^2_{a,h} \in (0,1), \tag{5}$$

then Assumption 1 also holds, as we now show. Given this per-period utility function, the definition of $X(z,\lambda)$ implies that for any good $\bar{l}$ and any two agents (a,h) and $(\bar{a},\bar{h})$ the following first order condition holds:

$$\lambda_{a,h} \cdot \zeta^2_{a,h} \cdot \left(\sum_l x^{\zeta^1}_{a,h,l} \right)^{\zeta^2_{a,h}-1} \cdot x^{\zeta^1-1}_{a,h,\bar{l}} = \lambda_{\bar{a},\bar{h}} \cdot \zeta^2_{\bar{a},\bar{h}} \cdot \left(\sum_l x^{\zeta^1}_{\bar{a},\bar{h},l} \right)^{\zeta^2_{\bar{a},\bar{h}}-1} \cdot x^{\zeta^1-1}_{\bar{a},\bar{h},\bar{l}}. \tag{6}$$

Considering Equation (6) for two different goods l and $\bar{l}$ and using the aggregate resource constraint we obtain the linear sharing rule, $x_{a,h,l} = x_{a,h,\bar{l}} \cdot \omega_l / \omega_{\bar{l}}$, which we plug in Equation (6) to get:

$$\lambda_{a,h} \cdot \zeta^2_{a,h} \cdot \left(\sum_l \left(\frac{\omega_l}{\omega_{\bar{l}}} \right)^{\zeta^1} \right)^{\zeta^2_{a,h}-1} \cdot x_{a,h,\bar{l}}^{\zeta^1 \zeta^2_{a,h}-1} = \lambda_{\bar{a},\bar{h}} \cdot \zeta^2_{\bar{a},\bar{h}} \cdot \left(\sum_l \left(\frac{\omega_l}{\omega_{\bar{l}}} \right)^{\zeta^1} \right)^{\zeta^2_{\bar{a},\bar{h}}-1} \cdot x_{\bar{a},\bar{h},\bar{l}}^{\zeta^1 \zeta^2_{\bar{a},\bar{h}}-1}. \tag{7}$$

Solving Equation (7) for $x_{\bar{a},\bar{h},\bar{l}}$, then summing over all agents $(\bar{a}, \bar{h})$, and finally using the aggregate resource constraint provides:

$$\omega^{\bar{l}} = \sum_{\bar{a},\bar{h}} \left(\frac{\lambda_{\bar{a},\bar{h}}}{\lambda_{a,h}} \right)^{\frac{1}{1-\zeta^1 \zeta^2_{\bar{a},\bar{h}}}} \left(\frac{\zeta^2_{\bar{a},\bar{h}}}{\zeta^2_{a,h}} \right)^{\frac{1}{1-\zeta^1 \zeta^2_{\bar{a},\bar{h}}}} \left(\sum_l \left(\frac{\omega_l}{\omega_{\bar{l}}} \right)^{\zeta^1} \right)^{\frac{\zeta^2_{\bar{a},\bar{h}}-1}{(\zeta^2_{a,h}-1)(1-\zeta^1 \zeta^2_{a,h})}} x_{a,h,\bar{l}}^{\frac{\zeta^1 \zeta^2_{a,h}-1}{\zeta^1 \zeta^2_{\bar{a},\bar{h}}-1}}. \tag{8}$$

Now let $\lambda \in \mathbb{R}^{(A-1)H}_{++}$, $\Delta\lambda \in \mathbb{R}^{(A-1)H}_{+} \setminus \{0\}$, and $\Delta\lambda_{a,h} = 0$. Applying Equation (8) to both λ and $\lambda + \Delta\lambda$ implies that $x_{a,h,\bar{l}}(\lambda) > x_{a,h,\bar{l}}(\lambda + \Delta\lambda)$ — the right hand side of (8) increases through the direct effect of increasing λ to $\lambda + \Delta\lambda$, which can only be offset by a decrease in $x_{a,h,\bar{l}}$. This shows the first part of Assumption 1, as $D_x u_{a,h}(X(z,\lambda)) \cdot X(z,\lambda) = \zeta^1 \zeta^2_{a,h} \left(\sum_l x^{\zeta^1}_{a,h,l} \right)^{\zeta^2_{a,h}}$ increases in $x_{a,h,\bar{l}}$. The second part of Assumption 1 is trivial. All in all, we have shown that the per-period utility function in (5) satisfies Assumption 1. We now prove existence of recursive equilibrium under this assumption.

THEOREM 1 *A recursive equilibrium exists if Assumption 1, and the assumptions on preferences, endowments, and dividends stated in Section 2 are satisfied.*

Under very weak assumptions on the Markov transition an Arrow–Debreu equilibrium always exists (see Florenzano et al. (2001) for a model with continuous shocks and Kubler and Polemarchakis (2004) for a model with finitely many shocks). For simplicity we take as given that for all initial $\tilde{\lambda} \in int(\Delta^{(A-1)H-1})$ an Arrow–Debreu equilibrium exists. This equilibrium need, however, not be recursive.

The key to showing existence of a recursive equilibrium is to show that for given $\tilde{\lambda}$ there must be a unique continuation. Put differently, existence of a recursive equilibrium can only fail if given some initial shock, z_0, there exist two distinct Arrow–Debreu equilibria with instantaneous Negishi weight processes $(\lambda^1_t) \neq (\lambda^2_t)$ that satisfy $\tilde{\lambda}^1(z^0) = \tilde{\lambda}^2(z^0)$. Suppose that there indeed exist two Arrow–Debreu equilibria $(\lambda^1_t, \lambda^2_t)$ with $\lambda^1(z^\tau) \neq \lambda^2(z^\tau)$ while $\tilde{\lambda}^1(z^\tau) = \tilde{\lambda}^2(z^\tau)$. W.l.o.g. we can take $z^\tau = z^0$. Define $\underline{\lambda}_{a,h}(z^t) = \min[\lambda^1_{a,h}(z^t), \lambda^2_{a,h}(z^t)]$ for all $(a,h) \in \mathbf{A}$ and all z^t – since both λ^i are $\mathcal{F}_t$-adapted so must be $\underline{\lambda}$. We will show that the process of instantaneous Negishi weights $(\underline{\lambda}_t)$ does not lead to a feasible consumption allocation, which contradicts the assumption that there exist two equilibria $(\lambda^1_t) \neq (\lambda^2_t)$ as characterized above. We need the following three lemmas.

Lemma 2 *For all z^t and for any $(a,h) \in \mathbf{A}$ and any $y \in \mathbf{F}$, it must be true that*

$$\underline{\lambda}_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \underline{\lambda}(z^t))) \cdot y \leq$$
$$\min \left[\lambda^1_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \lambda^1(z^t))) \cdot y, \lambda^2_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \lambda^2(z^t))) \cdot y \right],$$

where the inequality is strict whenever $\underline{\lambda}(z^t) \notin \{\lambda^1(z^t), \lambda^2(z^t)\}$.

Proof. W.l.o.g. take $\lambda^1_{a,h}(z^t) \leq \lambda^2_{a,h}(z^t)$, thus $\underline{\lambda}_{a,h}(z^t) = \lambda^1_{a,h}(z^t)$. Since $-D_x u_{a,h}(X_{a,h}(z,.)) \cdot y$ satisfies the strict gross-substitutes property, we have

$$\underline{\lambda}_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \underline{\lambda}(z^t))) \cdot y \leq \lambda^1_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \lambda^1(z^t))) \cdot y,$$

where the inequality is strict if $\underline{\lambda}(z^t) \neq \lambda^1(z^t)$. Moreover, define $\hat{\lambda}$ by $\hat{\lambda}_{a',h'} = \underline{\lambda}_{a',h'}(z^t)$ for $(a',h') \neq (a,h)$ and $\hat{\lambda}_{a,h} = \lambda^2_{a,h}(z^t)$. By the gross-substitutes property we must have

$$\begin{aligned} \lambda^2_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \lambda^2(z^t))) \cdot y &\geq \hat{\lambda}_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \hat{\lambda}(z^t)) \cdot y \\ &\geq \underline{\lambda}_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \underline{\lambda}(z^t))) \cdot y, \end{aligned}$$

where the second inequality follows from the fact that $\lambda_{a,h}(z^t) D_x u_{a,h}(X_{a,h}(z_t, \lambda(z^t)))$ is identical across all agents for any λ, in particular for $\hat{\lambda}$ and $\underline{\lambda}$. The first inequality holds strict if $\hat{\lambda}(z^t) \neq \lambda^2(z^t)$, which is the case if $\underline{\lambda}(z^t) \neq \lambda^2(z^t)$. □

Lemma 3 *For all z^τ and all (a,h),*

$$w_{1,h}(z^\tau; (\underline{\lambda}_t)) \geq \min \left[w_{1,h}(z^\tau; (\lambda^1_t)), w_{1,h}(z^\tau; (\lambda^2_t)) \right] = 0. \tag{9}$$

Proof. Applying Assumption 1 to

$$v_{A,h}(z^\tau; \underline{\lambda}_t) = \underline{\lambda}_{A,h}(z^\tau) D_x u_{A,h}(X_{A,h}(z_\tau, \underline{\lambda}(z^\tau))) \cdot (X_{A,h}(z_\tau, \underline{\lambda}(z^\tau)) - e_{a,h}(z_\tau)),$$

we find that the following is satisfied for $a = A$:

$$v_{a,h}(z^\tau; (\underline{\lambda}_t)) \geq \begin{cases} v_{a,h}(z^\tau; (\lambda^1_t)), & \text{if } \underline{\lambda}_{a,h}(z^\tau) = \lambda^1_{a,h}(z^\tau), \\ v_{a,h}(z^\tau; (\lambda^2_t)), & \text{if } \underline{\lambda}_{a,h}(z^\tau) = \lambda^2_{a,h}(z^\tau), \end{cases} \quad \text{for all } z^\tau \text{ and } h. \tag{10}$$

We now show that if (10) holds for $a+1$, then it also does for a. Suppose w.l.o.g. that $\underline{\lambda}_{a,h}(z^\tau) = \lambda^1_{a,h}(z^\tau)$. For each $z^{\tau+1} \succ z^\tau$ there are two cases possible. In the first case, $\lambda^1_{a+1,h}(z^{\tau+1}) = \delta_{a+1,h}(z_{\tau+1}) \lambda^1_{a,h}(z^\tau)$, then $\underline{\lambda}_{a+1,h}(z^{\tau+1}) = \lambda^1_{a+1,h}(z^\tau)$, and thus $v_{a+1,h}(z^{\tau+1}; (\underline{\lambda}_t)) \geq v_{a+1,h}(z^{\tau+1}; (\lambda^1_t))$ by the induction hypothesis. In the second case, $\lambda^1_{a+1,h}(z^{\tau+1}) > \delta_{a+1,h}(z_{\tau+1}) \lambda^1_{a,h}(z^\tau)$, then $v_{a+1,h}(z^{\tau+1}; (\underline{\lambda}_t)) \geq v_{a+1,h}(z^{\tau+1}; (\lambda^1_t)) = 0$. Summing up, we find that (10) holds for a.

Again suppose w.l.o.g. that $\underline{\lambda}_{a,h}(z^\tau) = \lambda^1_{a,h}(z^\tau)$. By (10) and Lemma 2, we have for all z^τ and h:

$$\begin{aligned} w_{a,h}(z^\tau; (\underline{\lambda}_t)) &= v_{a,h}(z^\tau; (\underline{\lambda}_t)) - \mathbb{E}_\tau \left[\sum_{i=0}^{A-a-1} \underline{\lambda}_{i,h}(z^{\tau+i}) D_x u_{a+i,h}(X_{a+i,h}(z_{\tau+i}, \underline{\lambda}(z^{\tau+i})) \cdot f_{a+i,h}(z_{\tau+i}) \right] \\ &\geq v_{a,h}(z^\tau; (\lambda^1_t)) - \mathbb{E}_\tau \left[\sum_{i=0}^{A-a-1} \lambda^1_{i,h}(z^{\tau+i}) D_x u_{a+i,h}(X_{a+i,h}(z_{\tau+i}, \lambda^1(z^{\tau+i})) \cdot f_{a+i,h}(z_{\tau+i}) \right] \\ &= w_{a,h}(z^\tau; (\lambda^1_t)). \end{aligned}$$

This finishes the proof. □

Finally we can state a stronger version of the lemma for the agents initially alive.

LEMMA 4 *For all* $(a, h) \in \mathbf{A}_{-1}$,

$$w_{a,h}(z^0; (\underline{\lambda}_t)) \geq \max\left[w_{a,h}(z^0; (\lambda_t^1)), w_{a,h}(z^0; (\lambda_t^2))\right], \tag{11}$$

with the inequality holding strict for some (a, h) *if* $\underline{\lambda}(z^0) \notin \{\lambda^1(z^0), \lambda^2(z^0)\}$.

Proof. As in Lemma 3 the value of f-endowments is smaller under $\underline{\lambda}$ then both under λ^1 and λ^2. It must be strictly smaller if f-endowments are positive and if $\underline{\lambda}(z^0) \notin \{\lambda^1(z^0), \lambda^2(z^0)\}$. It therefore suffices to prove that

$$v_{a,h}(z^0; (\underline{\lambda}_t)) \geq \max\left[v_{a,h}(z^0; (\lambda_t^1)), v_{a,h}(z^0; (\lambda_t^2))\right]. \tag{12}$$

Suppose w.l.o.g. that $\underline{\lambda}_{a,h}(z^0) = \lambda^1_{a,h}(z^0)$. Then (10), which we have shown in the proof of Lemma (2), implies

$$v_{a,h}(z^0; (\underline{\lambda}_t)) \geq v_{a,h}(z^0; (\lambda_t^1)).$$

To derive (12), note that $\lambda^2_{a,h}(z^0) > \lambda^1_{a,h}(z^0)$ only if

$$v_{a,h}(z^0; (\lambda_t^2)) = 0 \leq v_{a,h}(z^0; (\lambda_t^1)) \leq v_{a,h}(z^0; (\underline{\lambda}_t)). \tag{13}$$

Inequality (13) must hold strict for $\underline{\lambda}(z^0) \notin \{\lambda^1(z^0), \lambda^2(z^0)\}$ – if f-endowments are zero e-endowments must be positive since we assume positive endowments. □

The equilibrium conditions require that for both $i = 1, 2$

$$w_{1,h}(z^0; (\lambda_t^i)) = 0, \sum_{(a,h)\in\mathbf{A}_{-1}} w_{a,h}(z^0, (\lambda_t^i)) - q(z^0; (\lambda_t^i)) = 0,$$

where we define the cum dividend price of the tree by

$$q(z^0; (\lambda_t)) = \mathbb{E}_0\left[\sum_{\tau=0}^{\infty} \lambda_{1,1}(z^\tau) D_x u_{1,1}(X_{1,1}(z_\tau, \lambda(z^\tau))) d(z_\tau)\right].$$

We can thus use the above equilibrium conditions for $(\lambda_t^1, \lambda_t^2)$ and Lemmas 2 - 4 to show that for $\underline{\lambda}(z^0) \notin \{\lambda^1(z^0), \lambda^2(z^0)\}$:

$$\sum_{(a,h)\in\mathbf{A}} w_{a,h}(z^0, (\underline{\lambda}_t)) - q(z^0; (\underline{\lambda}_t)) + \mathbb{E}_0\left[\sum_{t=1}^{\infty} w_{1,h}(z^t; (\underline{\lambda}_t))\right] > 0, \tag{14}$$

Using that $\underline{\lambda}$ is summable, as λ^1 and λ^2 are, Equation (14) implies that under $\underline{\lambda}$ the present value of consumption exceeds the present value of endowments and dividends. Since this is a contradiction to feasibility of $(\underline{\lambda}_t)$, we have proven that there cannot be two continuation equilibria and thus there exists a recursive equilibrium.

Finally, we have to show that we must indeed have $\underline{\lambda}(z^0) \notin \{\lambda^1(z^0), \lambda^2(z^0)\}$: Suppose w.l.o.g. $\lambda^1_{a,h}(z^0) < \lambda^2_{a,h}(z^0)$ for some (a,h). Then there must be some other $(\bar{a}, \bar{h})$ with $\lambda^1_{\bar{a},\bar{h}}(z^0) > \lambda^2_{\bar{a},\bar{h}}(z^0)$. To see this, suppose not, thus $\lambda^1(z^0) \leq \lambda^2(z^0)$. Since some agents must be unconstrained in the equilibrium described by λ^2 we must have that $\lambda^1_{a,h}(z^0) = \lambda^2_{a,h}(z^0)$ for some (a,h). Depending on whether $a = 1$ or $a > 1$ the strict gross-substitutes property then implies either $w_{1,h}(z^0; (\lambda^2_t)) > 0$ for some h, or

$$\sum_{(a,h)\in\mathbf{A}_{-1}} w_{a,h}(z^0, (\lambda^2_t)) - q(z^0; (\lambda^2_t)) > 0,$$

and hence λ^2 cannot describe an equilibrium.

6 Existence with norm-continuous transition

In this section we focus on the version of the model where agents cannot borrow against future endowments, that is $f_{a,h}(z) = 0$ for all $(a,h) \in \mathbf{A}$ and all $z \in \mathbf{Z}$. This assumption simplifies the analysis considerably — it is subject to further research to extend the result to the case with some unsecured borrowing. Furthermore, and in addition to the assumptions made in Section 2, we assume that the exogenous state space $\mathbf{Z}$ can be decomposed into three complete, separable metric spaces $\mathbf{Z} = \mathbf{Y}_\delta \times \mathbf{Y}_e \times \tilde{\mathbf{Z}}$. For the Borel σ-algebras we then have $\mathcal{Z} = \mathcal{Y}_\delta \otimes \mathcal{Y}_e \otimes \tilde{\mathcal{Z}}$, and the state is given by $z = (y_\delta, y_e, \tilde{z})$. We denote the transition probability on $\mathbf{Z}$ by $\mathbb{P}_z(\cdot|z)$ and the marginal distributions by $\mathbb{P}_{y_\delta}(\cdot|z)$, $\mathbb{P}_{y_e}(\cdot|z)$, and $\mathbb{P}_{\tilde{z}}(\cdot|z)$. The first shock component, y_δ, affects discount factors continuously, ensuring norm-continuity of the transition and thereby the existence of a solution to the period-to-period problem. The second component, y_e, corresponds to the noise component in Duggan (2012) and allows us to cast the fixed point argument in terms of value functions that are averages over y_e. This trick ensures that the correspondence from value functions to updated value functions is convex, which is necessary to apply the Debreu-Fan-Glicksberg theorem. Finally, the third component, $\tilde{z}$, is the 'standard' shock component which has to satisfy only weak conditions. In particular, it might include discrete shocks. More precisely, we make the following assumptions about the three components of $\mathcal{Z}$:

- $y_\delta \in \mathbf{Y}_\delta$ influences discount factors, but neither endowments nor the exogenous transition. The conditional distribution of y'_δ given z and $\tilde{z}'$, denoted by $\mathbb{P}_{y_\delta}(\cdot|z, \tilde{z}')$, is absolutely continuous with respect to the Lebesgue measure μ on $(\mathbf{Y}_\delta, \mathcal{Y}_\delta)$, where $\mathbf{Y}_\delta \subset \mathbb{R}^{(A-1)H-1}$. The Radon-Nikodym derivative of $\mathbb{P}_{y_\delta}(\cdot|z, \tilde{z}')$ with respect to μ is denoted by $r_{y_\delta}(\cdot|z, \tilde{z}')$ and vanishes at the boundary of $\mathbf{Y}_\delta$.

- $y_e \in \mathbf{Y}_e$ influences endowments but not discount factors. Conditional on next period's $\tilde{z}'$, the distribution of y'_e does not depend on the current state z. The conditional distribution $\mathbb{P}_{y_e}(\cdot|\tilde{z}')$ is absolutely continuous with respect to an atomless probability measure ν on $(\mathbf{Y}_e, \mathcal{Y}_e)$, its

Radon-Nikodym derivative is $r_{y_e}(.|\tilde{z}')$.

- $\mathbb{P}_{\tilde{z}}(.|z)$ is absolutely continuous with respect to a (not necessarily non-atomic) probability measure $\tilde{\kappa}$ on $(\tilde{\mathbf{Z}}, \tilde{\mathcal{Z}})$, its Radon-Nikodym derivative is $r_{\tilde{z}}(\cdot|z)$.

Finally, we assume that individual endowments as well as dividends are bounded from below by $\underline{e} > 0$, and that discount factors $\delta : \mathbf{Y}_\delta \times \tilde{Z} \to D = [\underline{\delta}, \overline{\delta}]^{(A-1)H}$ satisfy the following two properties that are crucial to guarantee norm-continuity of the transition:

- For each $\tilde{z} \in \tilde{\mathbf{Z}}$, $\delta(\cdot, \tilde{z})$ is a diffeomorphism from $\mathbf{Y}_\delta$ to the image of $\mathbf{Y}_\delta$ under $\delta(\cdot, \tilde{z})$.
- For each $\tilde{z} \in \tilde{\mathbf{Z}}$, the function $\sum_{(a,h)\in\mathbf{A}_{-1}} \delta_{a,h}(\cdot, \tilde{z})$ is constant.

Our assumptions on utility and boundedness of endowments imply that there is a $\overline{E}$ such that for any $z \in \mathbf{Z}$ and any $(a, h) \in \mathbf{A}$

$$\sup_{\lambda\in\Delta^{AH-1}} D_x u_{a,h}(X_{a,h}(z, \lambda)) \cdot (X_{a,h}(z, \lambda) - e_{a,h}(z)) < \overline{E},$$

where $X(z, \lambda)$ is defined as in Equation (1) above . Our assumptions also imply that there exists a $\epsilon > 0$ such that for all $z \in \mathbf{Z}$, $(a, h) \in \mathbf{A}$, $\lambda \in \Delta^{AH-1}$, and $\lambda_{a,h} \leq \epsilon$:

$$D_x u_{a,h}(X_{a,h}(z, \lambda)) \cdot (X_{a,h}(z, \lambda) - e_{a,h}(z)) + \overline{\delta}^A \overline{E} < 0.$$

Using this $\epsilon > 0$, we define

$$\epsilon' = \epsilon \frac{\underline{\delta}}{AH\overline{\delta}}$$

and choose the endogenous state space to be

$$\mathbf{\Lambda} = \{(\tilde{\lambda}_{a,h})_{(a,h)\in\mathbf{A}_{-1}} \in \Delta^{(A-1)H-1} : \tilde{\lambda}_{a,h} \geq \epsilon' \text{ for all } (a, h) \in \mathbf{A}_{-1}\},$$

with a typical element $\tilde{\lambda} \in \mathbf{\Lambda}$. This definition ensures that if in the current period all budget constraints hold and all $v_{a,h} \geq 0$, then next period's endogenous state must lie in $\mathbf{\Lambda}$, because this period's $\tilde{\lambda} \geq \epsilon$. We define feasible γ to lie in a correspondence $\Gamma : \mathbf{\Lambda} \rightrightarrows \mathbb{R}^{AH}_+$ with

$$\Gamma(\tilde{\lambda}) = \{\gamma \in \mathbb{R}^{AH}_+ : \min_{(a,h)\in\mathbf{A}} \frac{\lambda_{a,h}}{\sum_{(a,h)\in\mathbf{A}} \lambda_{a,h}} \geq \epsilon\}$$

where

$$\lambda = \left((\gamma_{1h})_{h\in\mathbf{H}}, (\tilde{\lambda}_{a,h} + \gamma_{a,h})_{(a,h)\in\mathbf{A}_{-1}}\right).$$

THEOREM 2 *A recursive equilibrium exists under the stated assumptions on preferences, and on the stochastic process for endowments, dividends, and discount-factors.*

To prove Theorem 2, we follow Duggan's (2012) proof for the existence of a Markov-perfect equilibrium in stochastic games as close as possible. However, there are two important points where our general equilibrium model requires the proof to differ completely from the approach used for stochastic games. First, norm-continuity is not an assumption in our framework, but rests on assumptions about economic fundamentals, in particular on the shock to agents' discounting, y_δ. Second, the existence of a solution to the period-to-period problem with borrowing constraints uses existence results for complementarity problems. The following proof of Theorem 2 is detailed with regard to these two points yet concise when it comes to those parts where we can closely follow Duggan (2012).

As the shock component $y_\delta \in \mathbf{Y}_\delta$ is only transitory and does not influence endowments, it is not included in the state space and we let the state space be $\mathbf{S} = \mathbf{Y}_e \times \tilde{\mathbf{Z}} \times \mathbf{\Lambda}$ with Borel σ-algebra $\mathcal{S}$.[4] Furthermore, we define $\mathbf{Q} := \tilde{\mathbf{Z}} \times \mathbf{\Lambda}$ with Borel σ-algebra $\mathcal{Q}$ and typical element $q = (\tilde{z}, \tilde{\lambda})$. We can then decompose the state as $\mathbf{S} = \mathbf{Y}_e \times \mathbf{Q}$. Recall that the transition for the endogenous state is determined as follows: Given current state $s = (y_e, \tilde{z}, \tilde{\lambda})$ and 'action' γ, as well as next-period shocks $(y'_\delta, \tilde{z}')$, next period's recursive weights are given by

$$\tilde{\lambda}'_{\bar{a},\bar{h}} = \frac{\delta_{\bar{a},\bar{h}}(y'_\delta, \tilde{z}')\lambda_{\bar{a}-1,\bar{h}}}{\sum_{(a,h)\in\mathbf{A}_{-1}} \delta_{a,h}(y'_\delta, \tilde{z}')\lambda_{a-1,h}} \;\forall(\bar{a},\bar{h}) \in \mathbf{A}_{-1}, \text{with } \lambda = \Big((\gamma_{1h})_{h\in\mathbf{H}}, (\tilde{\lambda}_{a,h} + \gamma_{a,h})_{(a,h)\in\mathbf{A}_{-1}}\Big). \tag{15}$$

As y_δ does not influence the exogenous transition probability, we can define a transition probability $\mathbb{P}_q : \mathbf{Y}_e \times \tilde{\mathbf{Z}} \times \text{ graph}(\Gamma) \times \mathcal{Q} \to [0,1]$ by demanding for all $(y_e, \tilde{z}, \tilde{\lambda}, \gamma)$ and $B \in \mathcal{Q}$

$$\mathbb{P}_q\left(B \mid y_e, \tilde{z}, \tilde{\lambda}, \gamma\right) = \mathbb{P}_z\left(C|z\right),$$

where

$$C = \{z' \in Z : \exists\tilde{\lambda}' \text{ s.t. } (\tilde{z}', \tilde{\lambda}') \in B \text{ and Equation (15) holds}\}.$$

A necessary step for obtaining existence is the observation that the mapping $\gamma \to \mathbb{P}_q(.|s,\gamma)$ is 'norm-continuous', as Lemma 5 states.

Lemma 5 *For all* $s \in \mathbf{S}, \gamma \in \Gamma(\tilde{\lambda})$ *and each sequence* $\gamma^n \to \gamma$, $\gamma^n \in \Gamma(\tilde{\lambda})\ \forall n$, *the sequence* $\mathbb{P}_q(.|s,\gamma^n)$ *converges to* $\mathbb{P}_q(.|s,\gamma)$ *in total variation, that is*

$$\sup_{B\in\mathcal{Q}} \|\mathbb{P}_q(B|s,\gamma^n) - \mathbb{P}_q(B|s,\gamma)\| \to 0.$$

Proof. To express $\mathbb{P}_q(.|s,\gamma)$ as an integral over $B \subset \mathbf{Q} = \tilde{\mathbf{Z}} \times \mathbf{\Lambda}$ we need to find the density $r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma)$ of $\tilde{\lambda}'$ with respect to the Lebesque measure μ on Λ. For this purpose, consider the functions $g_{(z,\tilde{z}',\gamma)} : Y_\delta \to \Lambda$ with $g_{(z,\tilde{z}',\gamma)}(y_\delta) = \tilde{\lambda}$, where $\tilde{\lambda}$ is determined by Equation (15) given y_δ and $(z, \tilde{z}', \gamma)$. When its range is restricted to $\bar{\Lambda}_{(z,\tilde{z}',\gamma)} := g_{(z,\tilde{z}',\gamma)}(Y_\delta) \subset \Lambda$, this function is a

[4] To simplify the notation, we occasionally use $z = (y_\delta, y_e, \tilde{z})$ even when only $(y_e, \tilde{z})$ matters, e.g. in $X(z, \tilde{\lambda})$.

diffeomorphism, as $\delta(\cdot, \tilde{z}')$ is a diffeomorphism and $\sum_{(a,h)\in\mathbf{A}_{-1}} \delta_{a,h}(\cdot, \tilde{z}')$ is constant.[5] Then define

$$r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma) := \begin{cases} r_{y_\delta}(g^{-1}_{(z,\tilde{z}',\gamma)}(\tilde{\lambda}')|z, \tilde{z}') \cdot \left| J(g^{-1}_{(z,\tilde{z}',\gamma)}(\tilde{\lambda}')) \right| & \text{if } \exists\, y_\delta : g_{(z,\tilde{z}',\gamma)}(y_\delta) = \tilde{\lambda}' \\ 0 \quad \text{otherwise,} \end{cases}$$

where $|J(\cdot)|$ denotes the determinant of the Jacobian. With this definition of $r_{\tilde{\lambda}}$, we get that

$$\begin{aligned} \mathbb{P}_q(B|s,\gamma) &= \int_{\tilde{Z}} \int_{Y_\delta} \mathbf{1}_B \left[(\tilde{z}', g_{(z,\tilde{z}',\gamma)}(y'_\delta)\right] r_{y_\delta}(y'_\delta|z, \tilde{z}') r_{\tilde{z}}(\tilde{z}'|z) d\mu(y'_\delta) d\tilde{\kappa}(\tilde{z}') \\ &= \int_{\tilde{Z}} \int_{\bar{\Lambda}_{(z,\tilde{z}',\gamma)}} \mathbf{1}_B \left[(\tilde{z}', \tilde{\lambda}')\right] r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma) r_{\tilde{z}}(\tilde{z}'|z) d\mu(\tilde{\lambda}') d\tilde{\kappa}(\tilde{z}') \\ &= \int_{\tilde{Z}} \int_{\Lambda} \mathbf{1}_B \left[(\tilde{z}', \tilde{\lambda}')\right] r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma) r_{\tilde{z}}(\tilde{z}'|z) d\mu(\tilde{\lambda}') d\tilde{\kappa}(\tilde{z}') \\ &= \int_B r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma) r_{\tilde{z}}(\tilde{z}'|z) d\mu(\tilde{\lambda}') d\tilde{\kappa}(\tilde{z}'). \end{aligned}$$

where the second equality is implied by the change of variables theorem (see, e.g., Theorem 13.49 in Aliprantis and Border), and the third equality follows from the fact that $r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma)$ is zero on $\Lambda \setminus \bar{\Lambda}_{(z,\tilde{z}',\gamma)}$. By Scheffe's Lemma, the statement of Lemma 5 follows from the derived representation of $\mathbb{P}_q(B|s,\gamma)$, if for all $(z, \tilde{z}', \tilde{\lambda}')$: $r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma^n) \to r_{\tilde{\lambda}}(\tilde{\lambda}'|z, \tilde{z}', \gamma)$. This in turn follows from the definition of r_{y_δ} as $g^{-1}_{(z,\tilde{z}',\gamma)}(\tilde{\lambda}')$ considered as a function of γ is continuous and continuously differentiable (see Equation (15)), and $r_{y_\delta}(\cdot|z, \tilde{z}')$ is continuous and vanishes at the boundary. □

Let $\overline{\mathbf{V}}$ be the set of all measurable functions $\bar{V} : \mathbf{Q} \to \mathbb{R}_+^{(A-1)H}$ that are essentially bounded above by $\bar{\delta}^A \overline{E}$ and below by 0. These can be thought of as 'average' value functions over all realisations of y_e. They lie in the space $L^n_\infty(Q, \mathcal{Q}, \kappa)$ of essentially bounded and measurable (equivalence classes of) functions from $\mathbf{Q}$ to $\mathbb{R}^n$ with $n = (A-1)H$ and $\kappa := \tilde{\kappa} \otimes \mu$. We endow $L^n_\infty(Q, \mathcal{Q}, \kappa)$ with the weak* topology $\sigma(L^n_\infty, L^n_1)$. The set $\overline{\mathbf{V}}$ is then a non-empty, convex and compact subset of a locally convex, Hausdorff topological vector space (see Duggan (2012), and references therein). Given any $\bar{V} \in \overline{\mathbf{V}}$, we define for all $(a,h) \in \mathbf{A}$ and all (s,γ)

$$\mathbb{B}^{\overline{V}}_{(a,h)}(s,\gamma,\overline{V}) := D_x u_{a,h}(X(z, \hat{\lambda}+\gamma)) \cdot (X(z, \hat{\lambda}+\gamma) - e_{a,h}(z)) + \int_{q'} \delta(q', s, \gamma) \overline{V}_{a+1,h}(q') d\mathbb{P}_q(q'|s,\gamma),$$

where $\hat{\lambda}_{a,h} = \tilde{\lambda}_{a,h}\ \forall (a,h) \in \mathbf{A}_{-1}$, $\hat{\lambda}_{1,h} = 0\ \forall h \in \mathbf{H}$, and $q' = (\tilde{z}', \tilde{\lambda}')$, with $\tilde{\lambda}'$ given by Equation (15). Note that when integrating over q', we need to write the discount factor as depending on (q', s, γ). The reason is that y_δ is not included in the state space. Yet given (s,γ), there is a unique δ that is consistent with $q' = (\tilde{z}', \tilde{\lambda}')$ — as $\sum_{(a,h)\in\mathbf{A}_{-1}} \delta_{a,h}(y_\delta, \tilde{z}')$ does not depend on y_δ. The function $\delta(q', s, \gamma)$ is thus well defined. For the proof of Theorem 2 we will need the following properties of $\mathbb{B}^{\overline{V}}$.

LEMMA 6 *Given any $\overline{V} \in \overline{\mathbf{V}}$ and $\gamma \in \Gamma(\tilde{\lambda})$, the function $\mathbb{B}^{\overline{V}}(s,\gamma,\overline{V})$ is measurable in s. For given s the function is continuous in γ and $\overline{V}$.*

[5]Using the obvious embedding we now regard Λ as a positive-measure subset of $\mathbb{R}^{(A-1)H-1}$ rather than as a null set in $\mathbb{R}^{(A-1)H}$.

Proof. Let $(a,h) \in \mathbf{A}$ be arbitrary. We first show that $\mathbb{B}^{\overline{V}}_{a,h}(s,\gamma,\overline{V})$ is measurable in s. First, $D_x u_{a,h}(X(z,\hat{\lambda}+\gamma))\cdot(X(z,\hat{\lambda}+\gamma)-e_{a,h}(z))$ is continuous in $\hat{\lambda}$ and γ for fixed z and it is measurable in $(z,\tilde{\lambda})$ since by Theorem 18.19 in Aliprantis and Border (2006) $X(z,\lambda)$ is measurable in (z,λ) which is a continuous function of $(z,\tilde{\lambda})$. As y_δ does not influence $X(z,\lambda)$, measurability in $(z,\tilde{\lambda}) = (y_\delta, y_e, \tilde{z}, \tilde{\lambda})$ implies measurability in $s = (y_e, \tilde{z}, \tilde{\lambda})$. Second, $\int_{q'} \delta(q',s,\gamma)\overline{V}_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma)$ is measurable in s since the integrand is essentially bounded and $\mathbb{P}_q$ is a transition probability (see Theorem 19.7 in Aliprantis and Border (2006)). To prove continuity in $(\gamma,\overline{V})$, fix s and consider a sequence $\{\gamma^n,\overline{V}^n\}$ converging to $(\gamma,\overline{V})$. For all (a,h) we then have

$$\begin{aligned}
&\left|\int_{q'} \delta(q',s,\gamma^n)\overline{V}^n_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma^n) - \int_{q'} \delta(q',s,\gamma)\overline{V}_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma)\right| \\
\leq\ &\left|\int_{q'} \delta(q',s,\gamma^n)\overline{V}^n_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma^n) - \int_{q'} \delta(q',s,\gamma)\overline{V}^n_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma)\right| \\
&+\left|\int_{q'} \delta(q',s,\gamma)\overline{V}^n_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma) - \int_{q'} \delta(q',s,\gamma)\overline{V}_{a+1,h}(q')d\mathbb{P}_q(q'|s,\gamma)\right| \\
\leq\ &\overline{\delta}^{A+1}\overline{E}\,||\mathbb{P}_q(.|s,\gamma^n) - \mathbb{P}_q(.|s,\gamma)|| + \left|\int_{q'} \left(\overline{V}^m_{a+1,h}(q') - \overline{V}_{a+1,h}(q')\right)\delta(q',s,\gamma)r_q(q'|s,\gamma)d\kappa(q')\right| \to 0,
\end{aligned}$$

where $r_q(q'|s,\gamma) := r_{\tilde{\lambda}}(\tilde{\lambda}'|z,\tilde{z}',\gamma)r_{\tilde{z}}(\tilde{z}'|z)$. The first step of this derivation is simply the triangle inequality. For the second step we use that $\overline{V}^m \in \overline{\mathbf{V}}$ is essentially bounded by $\overline{\delta}^A\overline{E}$, and $\delta(q',s,\gamma)$ is bounded by $\overline{\delta}$. Finally, the first term goes to zero as $\mathbb{P}_q$ is norm-continuous, while the second term goes to zero because $\{\overline{V}^n\}$ converges to $\overline{V}$ in the weak* topology and $\delta(q',s,\gamma)r_q(q'|s,\gamma)$ lies in L_1.
□

The next lemma guarantees the existence of a policy that satisfies the equilibrium conditions, given a $\bar{V}$ next period.[6]

LEMMA 7 *For each s there exists a $\gamma \geq 0$ such that for all (a,h),*

$$\mathbb{B}^{\bar{V}}_{a,h}(s,\gamma,\bar{V}) \geq 0, \quad \mathbb{B}^{\bar{V}}_{a,h}(s,\gamma,\bar{V})\gamma_{a,h} = 0. \tag{16}$$

To prove the lemma, we use the following result about non-linear complementarity problems. The result follows directly from Theorem 1.4 in Kojima (1975).

LEMMA 8 *Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, suppose there exists a $k > 0$ such that for all $x \in \{x \in \mathbb{R}^n_+ : \sum_i x_i = k\}$ it holds that $\max_i x_i f_i(x) > 0$. Then there exists $\bar{x} \in \mathbb{R}^n_+$ such that $f(\bar{x}) \geq 0$ and $\bar{x}\cdot f(\bar{x}) = 0$.*

With this result we can now prove Lemma 7.

Proof of Lemma 7. Since $\bar{V} \geq 0$ and $(\hat{\lambda}_{a,h} + \gamma_{a,h}) > 0$ it suffices to show that there exists a set

[6]In our setup, this result plays the same role as the result that there always exists a mixed strategy Nash equilibrium for the stage game plays in the stochastic game setup.

$\{\gamma \in \mathbb{R}_+^{AH} : \sum_{(a,h)\in\mathbf{A}} \gamma_{a,h} = k\}$ as in Lemma 8 so that for all $\gamma \in K$

$$\max_{a,h}[\gamma_{a,h}(\hat{\lambda}_{a,h}+\gamma_{a,h})D_x u_{a,h}(X_{a,h}(z,\hat{\lambda}+\gamma))\cdot(X_{a,h}(z,\hat{\lambda}+\gamma)-e_{a,h}(z))] > 0.$$

Since by definition of $X(\cdot)$, $(\hat{\lambda}_{a,h}+\gamma_{a,h})D_x u_{a,h}(X_{a,h}(z,\hat{\lambda}+\gamma))$ is identical across agents, and since the tree pays strictly positive dividends, we must have that

$$\sum_{(a,h)\in\mathbf{A}} (\hat{\lambda}_{a,h}+\gamma_{a,h})D_x u_{a,h}(X_{a,h}(z,\hat{\lambda}+\gamma))\cdot(X_{a,h}(z,\hat{\lambda}+\gamma)-e_{a,h}(z)) > 0.$$

For sufficiently large k, it is easy to see that whenever $\gamma_{a,h}=0$ we have $X_{a,h}(z,\hat{\lambda}+\gamma) < e_{a,h}(z)$ and thus

$$(\hat{\lambda}_{a,h}+\gamma_{a,h})D_x u_{a,h}(X_{a,h}(z,\hat{\lambda}+\gamma))\cdot(X_{a,h}(z,\hat{\lambda}+\gamma)-e_{a,h}(z)) < 0.$$

This implies that

$$\max_{a,h}[\gamma_{a,h}(\hat{\lambda}_{a,h}+\gamma_{a,h})D_x u_{a,h}(X_{a,h}(z,\hat{\lambda}+\gamma))\cdot(X_{a,h}(z,\hat{\lambda}+\gamma)-e_{a,h}(z))] > 0. \quad \Box$$

For the proof, it is crucial that we can restrict $\bar{V}$ to be non-negative. It is subject to further researcher to examine the case of positive f-endowments where $W=V$ no longer holds and one has to keep track of W-functions that can become negative.
Note that there is a compact set $\overline{\Gamma} \subset \mathbb{R}_+^{AH}$ so that the solution in the above lemma always lies in this set, $\gamma \in \overline{\Gamma}$. The fact that γ is bounded above follows from our assumptions on utility and the fact that there must always be at least one unconstrained agent.

We can define non-empty correspondences $s \to \mathbf{N}_{\bar{V}}(s)$ to contain all γ that solve Equation (16) and $s \to \mathbf{P}_{\bar{V}}(s)$ by $\mathbf{P}_{\bar{V}}(s) = \{\mathbb{B}^{\bar{V}}(s,\gamma,\bar{V})$ for $\gamma \in \mathbf{N}_{\bar{V}}(s)\}$. Recall from Definition 18.1 in Aliprantis and Border (2006) that a correspondence $g : \mathbf{Z} \rightrightarrows \mathbb{R}^n$ is called *measurable* if for any closed set $\mathbf{G} \subset \mathbb{R}^n$ it holds that $\{z \in \mathbf{Z} : g(z) \cap \mathbf{G} \neq \emptyset\} \in \mathcal{Z}$.

LEMMA 9 *Suppose $f(z,x) : \mathbf{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in z and continuous in x. Define a correspondence $g : \mathbf{Z} \rightrightarrows \mathbb{R}^n$ by*

$$g(z) = \{x \geq 0 : f(z,x) \geq 0 \text{ and } x \cdot f(z,x) = 0\}.$$

If $g(z)$ is uniformly bounded for all z, then it is measurable.

Proof. It is well known that a non-linear complementarity problem can be rewritten as a zero of a system of equations. Namely,

$$x \geq 0, f(z,x) \geq 0 \text{ and } x \cdot f(z,x) = 0$$

is equivalent to the existence of $(x,y) \geq 0$ such that

$$F(z,x,y) := \begin{pmatrix} Xy \\ y - f(z,x) \end{pmatrix} = 0, \text{ where } X = \mathrm{diag}(x).$$

Therefore, for any $x \in g(z)$ there is a y so that (x, y) are zeros of a system of continuous functions. Corollary 18.8 of Aliprantis and Border (2006) shows that the correspondence mapping z to $\{(x, y) \in K : F(z, x, y) = 0\}$, with K compact, is measurable. Since projection is a continuous function, the correspondence $g(.)$ must be measurable. $\square$

LEMMA 10 *For each $\bar{V} \in \overline{\mathbf{V}}$ the correspondences $\mathbf{N}_{\bar{V}}(s)$ and $\mathbf{P}_{\bar{V}}(s)$ are measurable.*

Proof. By Lemma 9 the correspondence $s \to \mathbf{N}_{\bar{V}}(s)$ must be measurable. Since $\mathbf{P}_{\bar{V}}$ is the image of $\mathbf{N}$ under a continuous function it is measurable as well. $\square$

Lemma 10 is analogous to Lemma 2 in Duggan (2012). Note, however, that instead of measurability, Duggan uses the weaker concept of lower measurability, which is sufficient to obtain all of his subsequent results. In contrast, we can obtain measurability of $\mathbf{N}_{\bar{V}}(s)$ and $\mathbf{P}_{\bar{V}}(s)$ by Lemma 9 which states that the correspondence from the state to the solutions of the complementarity problem is measurable. Following Duggan, we now define for each $y_e \in \mathbf{Y}_e$, $\Phi_{\bar{V}}(y_e)$ as the set of measurable selections of $\mathbf{P}_{\bar{V}}$ weighted with the density $r_{y_e}(y_e|\tilde{z})$. More precisely, for $f \in L_1^n$ we have $f \in \Phi_{\bar{V}}(y_e)$ if and only if for κ-almost all $q \in Q$, $f(q) \in r_{y_e}(y_e|\tilde{z})\mathbf{P}_{\bar{V}}(q, y_e)$. Then $\Phi : \mathbf{Y}_e \rightrightarrows L_1^n$ assigns to each possible value of y_e a set of functions in L_1^n. Integrating over y_e (using the Bochner integral, Definition 11.42 of Aliprantis and Border (2006)) one obtains a set $\mathbf{E}_{\bar{V}} \subset \overline{\mathbf{V}}$ defined as

$$\mathbf{E}_{\bar{V}} = \{\int \phi(y_e) d\nu(y_e) : \phi \text{ is a Bochner integrable selection of } \Phi\}.$$

Lemmas 3-5 in Duggan establish that for any $\bar{V}$ this is a non-empty, convex subset of $\overline{\mathbf{V}}$. The convex-valuedness is remarkable since there is no direct way to apply Lyapunov's theorem to infinite dimensional spaces. However, one can exploit the product structure of the problem and apply Lyapunov's theorem for each q separately and then combine these, using a theorem of Artstein (1989) to ensure measurability. Finally, Lemmas 6-7 in Duggan show that the correspondence $\bar{V} \to \mathbf{E}_{\bar{V}}$ has closed graph. All in all, we thus have a correspondence from $\overline{\mathbf{V}}$ to $\overline{\mathbf{V}}$ that must have a fixed point $\overline{V}^* \in \mathbf{E}_{\overline{V}^*}$ by the Kakutani-Fan-Glicksberg fixed point theorem (Corollary 17.55 in Aliprantis and Border (2006)). From this fixed point, we construct a recursive equilibrium as follows. There must exist a ν-integrable function $\phi : \mathbf{Y}_e \to \mathbb{R}_+^{AH}$ such that $\bar{V}^* = \int \phi(y_e) d\nu(y_e)$. Together with Theorem 11.47 of Aliprantis and Border (2006) this implies that there exists a $\kappa \otimes \nu$ -integrable function F with $F(s) \in r_{y_e}(y_e|\tilde{z})\mathbf{P}_{\bar{V}^*}(q, y_e)$ such that for κ-almost all q, $\bar{V}^*(q) = \int F(q, y_e) d\nu(y_e)$. The existence of a measurable function $\gamma^*(\cdot)$ is then guaranteed by Theorem 18.17 in Aliprantis and Border (2006). These functions constitute a recursive equilibrium.

Finally, we would like to make two remarks about the presented proof. First, the proof strategy does not require the OLG structure of the model and could thus also be used to prove existence in a version of Chien and Lustig (2010) with a finite number of agents. Second, recently He and Sun (2013) proved existence of stationary Markov perfect equilibria in stochastic games under a substantially more general assumption than the noise structure of Duggan (2012). Nevertheless, we follow Duggan (2012), because his assumption has a more straightforward economic interpretation.

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