Recursive equilibria in dynamic economies with stochastic production*

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Abstract

In this paper we prove the existence of recursive equilibria in a dynamic stochastic model with infinitely lived heterogeneous agents, several commodities, and general inter- and intra-temporal production. We illustrate the usefulness of our result by providing sufficient conditions for the existence of recursive equilibria in heterogeneous agent versions of both the Lucas asset pricing model and the neoclassical stochastic growth model.

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1 Introduction

The use of so-called recursive equilibria to analyze dynamic stochastic general equilibrium models has become increasingly important in financial economics, in macroeconomics, and in public finance. These equilibria are characterized by a pair of functions: a transition function mapping this period's "state" into probability distributions over next period's state, and a "policy function" mapping the current state into current prices and choices (see, e.g., Ljungquist and Sargent (2004) for an introduction). In applications that consider dynamic stochastic economies with heterogeneous agents and production, it is typically the current exogenous shock together with the capital stock and the beginning-of-period distribution of assets across individuals that define this recursive state. We will refer to recursive equilibria with this minimal "natural" state space simply as recursive equilibria, or—following the terminology of stochastic games—as stationary Markov equilibria. Unfortunately, for models with infinitely lived agents and incomplete financial markets no sufficient conditions for the existence of these stationary Markov equilibria can be found in the existing literature. In this paper we close this gap in the literature and prove the existence of recursive equilibria for a general class of stochastic dynamic economies with heterogeneous agents and production. To do so, we assume that there are two atomless shocks that are stochastically independent (conditional on a possible third shock that can be arbitrary). The first shock is purely transitory and only affects fundamentals that influence the endogenous state, while the second does not affect these fundamentals. We illustrate the usefulness of our results by providing sufficient conditions for the existence of recursive equilibria in heterogeneous agent versions of both the Lucas asset pricing model and the neoclassical stochastic growth model.

There are a variety of reasons for focusing on stationary Markov equilibria. Most importantly, recursive methods can be used to approximate stationary Markov equilibria numerically. Heaton and Lucas (1996), Krusell and Smith (1998), and Kubler and Schmedders (2003) are early examples of papers that approximate stationary Markov equilibria in models with infinitely lived, heterogeneous agents. Although an existence theorem for stationary Markov equilibria has not been available, applied research—even if explicitly aware of the problem—needs to focus on such equilibria, as there are no efficient algorithms for the computation of non-recursive equilibria.[1] For the case of dynamic games, Maskin and Tirole (2001) list several conceptual arguments in favor of stationary Markov equilibria. Duffie et al. (1994) give similar arguments that also apply to dynamic general equilibrium models: As prices vary across date events in a dynamic stochastic market economy, it is important that the price process is simple—for instance, Markovian on some minimal state space—to justify the assumption that agents have rational expectations.

[1] While Feng et al. (2013) provide an algorithm for this case, their method can only be used for very small-scale models.

Unfortunately, due to the non-uniqueness of continuation equilibria, stationary Markov equilibria do not always exist. This problem was first illustrated by Hellwig (1983) and since then has been demonstrated in different contexts. Kubler and Schmedders (2002) give an example showing the nonexistence of stationary Markov equilibria in models with incomplete asset markets and infinitely lived agents. Santos (2002) provides examples of nonexistence for economies with externalities. Kubler and Polemarchakis (2004) present such examples for overlapping generations (OLG) models, one of which we modify to fit our framework with infinitely lived agents and production, thereby demonstrating the possibility of nonexistence and motivating our analysis.

The existence of competitive equilibria for general Markovian exchange economies is shown in Duffie et al. (1994). The authors also prove that the equilibrium process is a stationary Markov process. However, we follow the well established terminology in dynamic games and do not refer to these equilibria as stationary Markov equilibria, because the state also contains consumption choices and prices from the previous period.

Citanna and Siconolfi (2010 and 2012) provide sufficient conditions for the generic existence of stationary Markov equilibria in OLG models. However, their arguments cannot be extended to models with infinitely lived agents or to models with occasionally binding constraints on agents' choices, and for their argument to work in their OLG framework they need to assume a very large number of heterogeneous agents within each generation.

Duggan (2012) and He and Sun (2017) give sufficient conditions for the existence of stationary Markov equilibria in stochastic games with uncountable state spaces. Building on work by Nowak and Raghavan (1992), He and Sun (2017) use a result from Dynkin and Evstigneev (1977) to provide sufficient conditions for the convexity of the conditional expectation operator. They show that the assumption of a public coordination device ("sunspot") in Nowak and Raghavan (1992) can be generalized to natural assumptions on the exogenous shock to fundamentals.

To show the existence of a recursive equilibrium, we characterize it by a function that maps the recursive state into the marginal utilities of all agents. Our first proposition shows that such a function describes a recursive equilibrium if it is a fixed point of an operator that captures the period-to-period equilibrium conditions. Using this characterization, we proceed in two steps to prove the existence of a recursive equilibrium. First, we make direct assumptions on the function that maps the current recursive state and current actions into the probability distribution of next period's recursive state. Assuming that this function varies continuously with current actions (a "norm-continuous" transition), the operator defined by the equilibrium conditions is a non-empty correspondence on the space of marginal utility functions. Unfortunately, the Fan–Glicksberg fixed-point theorem only guarantees the existence of a fixed point in the convex hull of this correspondence.

However, following He and Sun (2017) we give conditions that ensure that this is also a fixed point of the original correspondence. For this, we assume that the density of the transition probability is measurable with respect to a sigma algebra that is sufficiently coarse relative to the sigma algebra representing the total information available to agents. This establishes Proposition 2, which provides a first set of sufficient conditions for the existence of recursive equilibria. In a second step, we provide concrete assumptions on the stochastic process of exogenous shocks; assumptions that guarantee that the conditions of Proposition 2 are indeed satisfied. In particular, we assume that the shock process driving fundamentals contains, in addition to a possible *main component* that is not subject to specific assumptions, two components that both have an atomless distribution—the *transition component* and the *noise component*. The transition component is purely transitory and only affects fundamentals that influence the endogenous state. The noise component, in contrast, does not affect these fundamentals and is, conditional on the main component, independent of the transition component and of the previous period's shocks. Theorem 1 states that under these assumptions a recursive equilibrium exists.

We apply our result to two concrete models used frequently in macroeconomics and finance. We first prove the existence of a recursive equilibrium for a heterogeneous agent version of the Lucas (1978) asset pricing model with displacement risk. Second, we prove existence in a version of the Brock and Mirman (1972) stochastic growth model with inelastic labor supply and heterogeneous agents.

We present our main result and our two applications for models without short-lived financial assets—this makes the argument simpler and highlights the economic assumptions necessary for our existence result. As an extension, we introduce financial securities together with collateral constraints. In order to define a compact endogenous state space we need to make relatively strong assumptions on endowments and preferences, and to impose constraints on trades. It is subject to further investigation whether these assumptions can be relaxed. While it is well understood that without occasionally binding constraints on trade the existence of a recursive equilibrium cannot be established (see, e.g., Krebs (2004)), the assumptions made in this paper are certainly stronger than needed.

In a stationary Markov equilibrium the relevant state space consists of both endogenous and exogenous variables that are payoff-relevant,[2] predetermined, and sufficient for the optimization of individuals at every date event. There are several computational approaches that use individuals' "Negishi weights" as an endogenous state instead of the distribution of assets (see, e.g., Dumas and Lyasoff (2012) or Brumm and Kubler (2014)). Brumm and Kubler (2014) prove existence in a model with overlapping generations, complete financial markets, and borrowing constraints, but the

[2] Maskin and Tirole (2001) give a formal definition of payoff-relevant states for Markov equilibria in games.

approach does not extend to models with incomplete markets. In this paper we focus on equilibria that are recursive on the "natural" state space—that is to say, the space consisting of the exogenous shock and the asset holdings of all agents.

The rest of the paper is organized as follows: Section 2 presents the general model and gives an example in which no recursive equilibrium exists. Section 3 provides our existence theorem. Section 4 presents two applications. Detailed proofs can be found in the appendix.

2 A general dynamic Markovian economy

In this section we describe the economic model and define recursive equilibrium. While we consider an abstract and general model of a production economy, there are two special cases of the model that play an important role in practice. In the first, a heterogeneous agent version of the Lucas (1978) asset pricing model, agents trade in several long-lived assets that are in unit net supply and pay exogenous positive dividends in terms of the single consumption good. In the second, a version of the Brock–Mirman stochastic growth model with heterogeneous agents, there is a single capital good that can be used in intraperiod production, together with labor, to produce the single consumption good. This good can be consumed or stored in a linear technology yielding one unit of the capital good in the subsequent period. We show in Section 4 how our general existence proof can be used to provide sufficient conditions for existence of recursive equilibria in versions of these two models.

2.1 The model

Time is indexed by $t \in \mathbb{N}_0$. Exogenous shocks z_t realize in a complete, separable metric space $\mathbf{Z}$, and follow a first-order Markov process with transition probability $\mathbb{P}(.|z)$ defined on the Borel σ-algebra $\mathcal{Z}$ on $\mathbf{Z}$—that is, $\mathbb{P} : \mathbf{Z} \times \mathcal{Z} \to [0, 1]$. Let $(z_t)_{t=0}^{\infty}$, or in short (z_t), denote this stochastic process and let $(\mathcal{F}_t)$ denote its natural filtration (i.e., the smallest filtration such that (z_t) is $\mathcal{F}_t$-adapted). A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, \ldots, z_t)$ and called a date event. Whenever convenient, we simply use t instead of z^t.

We consider a production economy with infinitely lived agents. There are H types of agents, $h \in \mathbf{H} = \{1, \ldots, H\}$. At each date event there are L perishable commodities, $l \in \mathbf{L} = \{1, ..., L\}$, available for consumption and production. The individual endowments are denoted by $\omega_h(z^t) \in \mathbb{R}^L_+$ and we assume that they are time-invariant and measurable functions of the current shock. We take the consumption space to be the space of $\mathcal{F}_t$-adapted and essentially bounded processes. Each agent h has a time-separable expected utility function

$$U_h((x_{h,t})_{t=0}^{\infty}) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \delta^t u_h \left(z_t, x_{h,t} \right) \right],$$

where $\delta \in \mathbb{R}$ is the discount factor, $x_{h,t} \in \mathbb{R}^L_+$ denotes the agent's (stochastic) consumption at date t, and $(x_{h,t})_{t=0}^{\infty}$ denotes the agent's entire consumption process.

It is useful to distinguish between intertemporal and intraperiod production. Intraperiod production is characterized by a measurable correspondence $\mathbf{Y} : \mathbf{Z} \rightrightarrows \mathbb{R}^L$, where a production plan $y \in \mathbb{R}^L$ is feasible at shock z if $y \in \mathbf{Y}(z)$. For simplicity (and without loss of generality) we assume throughout that each $\mathbf{Y}(z)$ exhibits constant returns to scale so that ownership does not need to be specified.

Intertemporally each type $h = 1, ..., H$ has access to J linear storage technologies, $j \in \mathbf{J} = \{1, ..., J\}$. At a node z each technology (h, j) is described by a column vector of inputs $a^0_{hj}(z) \in \mathbb{R}^L_+$, and a vector-valued random variable of outputs in the subsequent period, $a^1_{hj}(z') \in \mathbb{R}^L_+$, $z' \in \mathbf{Z}$. We write $A^0_h(z) = (a^0_{h1}(z), \dots, a^0_{hJ}(z))$ for the $L \times J$ matrix of inputs and $A^1_h(z') = (a^1_{h1}(z'), \dots, a^1_{hJ}(z'))$ for the $L \times J$ matrix of outputs. We denote by $\alpha_h(z^t) = (\alpha_{h1}(z^t), ..., \alpha_{hJ}(z^t))^\top \in \mathbb{R}^J_+$ the levels at which the linear technologies are operated at node z^t by agent h.

Each period there are complete spot markets for the L commodities; we denote prices by $p(z^t) = (p_1(z^t), ..., p_L(z^t))$, a row vector. For what follows it will be useful to define the set of stored commodities (or "capital goods") to be

$$\mathbf{L}^K = \{l \in \mathbf{L} : \sum_{h \in \mathbf{H}} \sum_{j \in \mathbf{J}} a^1_{hjl}(z) > 0 \text{ for some } z \in \mathbf{Z}\},$$

and to define $\mathbf{K}^U = \{x \in \mathbb{R}^{HL}_+ : x_{hl} = 0 \text{ whenever } l \notin \mathbf{L}^K, h \in \mathbf{H}\}$. We decompose individual endowments into capital goods, f_h, and consumption goods, e_h, and define

$$f_{hl}(z) = \begin{cases} \omega_{hl}(z) & \text{if } l \in \mathbf{L}^K \\ 0 & \text{otherwise,} \end{cases}$$

and $e_h(z) = \omega_h(z) - f_h(z)$.

At $t = 0$ agents have some initial endowment in the capital goods that might be larger than $f_h(z_0)$ and to simplify notation we write the difference as $A^1_h(z_0)\alpha_h(z^{-1})$ for each agent h.

Given initial conditions $(f_h(z_0) + A^1_h(z_0)\alpha_h(z^{-1}))_{h \in \mathbf{H}} \in \mathbf{K}^U$, we define a *sequential competitive equilibrium* to be a process of $\mathcal{F}_t$-adapted prices and choices,

$$\left(\bar{p}_t, (\bar{x}_{h,t}, \bar{\alpha}_{h,t})_{h \in \mathbf{H}}, \bar{y}_t\right)_{t=0}^{\infty},$$

such that markets clear and agents optimize—that is to say, (A), (B), and (C) hold.

(A) Market clearing:

$$\sum_{h \in \mathbf{H}} (\bar{x}_h(z^t) + A^0_h(z_t)\bar{\alpha}_h(z^t) - \omega_h(z_t) - A^1_h(z_t)\bar{\alpha}_h(z^{t-1})) = \bar{y}(z^t), \quad \text{for all } z^t.$$

(B) Profit maximization:

$$\bar{y}(z^t) \in \arg \max_{y \in \mathbf{Y}(z_t)} \bar{p}(z^t) \cdot y.$$

(C) Each agent $h \in \mathbf{H}$ maximizes utility:

$$(\bar{x}_{h,t}, \bar{\alpha}_{h,t})_{t=0}^{\infty} \in \arg \max_{(x_{h,t}, \alpha_{h,t})_{t=0}^{\infty} \geqslant 0} U_h((x_{h,t})_{t=0}^{\infty})$$

$$\text{s.t. } \bar{p}(z^t) \cdot \left(x_h(z^t) + A_h^0(z_t)\alpha_h(z^t) - \omega_h(z_t) - A_h^1(z_t)\alpha_h(z^{t-1})\right) \leqslant 0, \quad \text{for all } z^t.$$

2.2 Recursive equilibrium

We take as an endogenous state variable the beginning-of-period holdings in capital goods, obtained from storage and as endowments. We fix an endogenous state space $\mathbf{K} \subset \mathbf{K}^U$ and take $\mathbf{S} = \mathbf{Z} \times \mathbf{K}$. A *recursive equilibrium* consists of "policy" and "pricing" functions

$$F_\alpha : \mathbf{S} \to \mathbb{R}_+^{HJ}, \quad F_x : \mathbf{S} \to \mathbb{R}_+^{HL}, \quad F_p : \mathbf{S} \to \Delta^{L-1}$$

such that for all initial shocks $z_0 \in \mathbf{Z}$, and all initial conditions $\left(A_h^1(z_0)\alpha_h(z^{-1}) + f_h(z_0)\right)_{h \in \mathbf{H}} \in \mathbf{K}$, there exists a competitive equilibrium

$$\left(\bar{p}_t, (\bar{x}_{h,t}, \bar{\alpha}_{h,t})_{h \in \mathbf{H}}, \bar{y}_t\right)_{t=0}^{\infty}$$

such that for all z^t

$$s(z^t) = \left(z_t, \left(A_h^1(z_t)\bar{\alpha}_h(z^{t-1}) + f_h(z_t)\right)_{h \in \mathbf{H}}\right) \in \mathbf{Z} \times \mathbf{K}$$

and $\bar{p}(z^t) = F_p(s(z^t))$, $\bar{x}(z^t) = F_x(s(z^t))$, $\bar{\alpha}(z^t) = F_\alpha(s(z^t))$.

For computational convenience one typically wants $\mathbf{K}$ to be convex—this will be guaranteed in our existence proof below but for now we do not include the requirement in the definition of recursive equilibrium. Note also that we chose the endogenous state space $\mathbf{K}$ to be a subset of $\mathbf{K}^U$, where $\mathbf{K}^U$ represents the holding of broadly defined capital goods $\mathbf{L}^K$. At the cost of notational inconvenience one could define capital goods and the space of capital holdings agent-wise by

$$\mathbf{L}_h^K = \{l \in \mathbf{L} : \sum_{j \in \mathbf{J}} a_{hjl}^1(z) > 0\}, \; \mathbf{K}_h^U = \{x \in \mathbb{R}_+^{HL} : x_l = 0 \text{ whenever } l \notin \mathbf{L}_h^K\}.$$

The endogenous state space would then satisfy $\mathbf{K} \subset \bigtimes_{h \in H} \mathbf{K}_h^U$, which could be considerably smaller than in the above definition, depending on the application. Similarly, one could make the space of capital holdings depend on the shock $z \in \mathbf{Z}$.

2.3 Possible nonexistence

Before we turn to our existence proof in Section 3, we now provide an example that illustrates why recursive equilibria may fail to exist. The example is inspired by Kubler and Polemarchakis (2004) and has the advantage that it can be analyzed analytically and all calculations are extremely simple.[3] In this example, agents make the same storage decisions in two different exogenous states.

[3] Kubler and Polemarchakis (2004) provide a second example where preferences and endowments are more standard, but we would need tools from computational algebraic geometry to analyze it and the basic point can be illustrated well in the simpler setup.

Yet these decisions are only consistent with intertemporal optimization because expectations about the next period's prices differ. Therefore, equilibrium prices are not only a function of capital holdings, but also of the previous period's exogenous state. Thus, an equilibrium that is recursive in the natural state does not exist.

The details of the example are as follows. We assume that there are only three possible shock realizations, $z' \in \{1,2,3\}$, which are independent of the current shock and equiprobable, thus $\pi(z'|z) = 1/3$ for all $z, z' \in \{1,2,3\}$. There are two commodities and two types of agent. As in Section 2.1, we assume that each agent maximizes time-separable expected utility, and to make computations simple we assume $\delta = 1/2$. Each agent has access to one storage technology.[4] Agent 1's technology transforms one unit of commodity 1 at given shocks $z = 1$ and $z = 2$ to one unit of commodity 1 in the subsequent period whenever shock 3 occurs. Agent 2's technology transforms one unit of commodity 2 at given shocks $z = 1$ and $z = 2$ to one unit of commodity 2 in the subsequent period whenever shock 3 occurs. At shock $z = 3$ no storage technology is available.[5] All in all, we have

$$\begin{aligned} a_1^0(1) = a_1^0(2) = (1,0), a_1^0(3) = \infty, \qquad & a_2^0(1) = a_2^0(2) = (0,1), a_2^0(3) = \infty \\ a_1^1(1) = a_1^1(2) = 0, a_1^1(3) = (1,0), \qquad & a_2^1(1) = a_2^1(2) = 0, a_2^1(3) = (0,1). \end{aligned}$$

We assume that the Bernoulli utility functions of agents 1 and 2 are as follows:

$$u_1(z = 1, (x_1, x_2)) = u_1(z = 2, x) = -\frac{1}{6x_1}, \quad u_1(z = 3, x) = -\frac{1}{x_1} + x_2,$$

$$u_2(z = 1, (x_1, x_2)) = u_2(z = 2, x) = -\frac{1}{6x_2}, \quad u_2(z = 3, x) = x_1 - \frac{1}{x_2}.$$

Endowments of agents of type 1 are

$$\omega_1(z = 1) = (\omega_{11}(1), \omega_{12}(1)) = (2,0), \quad \omega_1(z = 2) = (0.1, 0), \quad \omega_1(z = 3) = (0, 2),$$

and endowments of agents of type 2 are

$$\omega_2(z = 1) = (0, 0.1), \quad \omega_2(z = 2) = (0, 2), \quad \omega_2(z = 3) = (2, 0).$$

For simplicity we set up the example completely symmetrically. In shocks 1 and 2, agent 1 only derives utility from consumption of good 1 and is only endowed with good 1, agent 2 only derives utility from good 2 and is only endowed with this good.

It is easy to see that at shocks 1 and 2 there will never be any trade. By assumption, if shock 3 occurs there cannot be any storage. Therefore, the economy decomposes into one-period and

[4] To simplify notation we assume that each agent has his or her own technology but given our assumptions on endowments, below, it would be equivalent to assume that each agent has access to both technologies.

[5] The assumption is made for convenience—all one needs is productivity low enough to guarantee that the technology is not used. In a slight abuse of notation we write $a_h^0(3) = \infty$.

two-period "sub-economies". The only non-trivial case is when shock 3 is preceded by either shock 1 or 2. In these two-period economies, agents make a savings decision in the first period and interact in spot markets in the second period.

To analyze the equilibria in these two-period economies, it is useful to compute the individual demands in the second period in shock 3 as functions of the price ratio $\tilde{p} = \frac{p_2(z'=3)}{p_1(z'=3)}$ given amounts of commodity 1 obtained by agent 1's storage, κ_1, and amounts of commodity 2 obtained by agent 2's storage, κ_2. We obtain for agent 1,

$$x_1(\tilde{p}|\kappa) = \begin{cases} (\tilde{p}\omega_{12}(3) + \kappa_1, 0) & \text{for } \tilde{p}\omega_{12}(3) - \sqrt{\tilde{p}} + \kappa_1 \leqslant 0 \\ (\sqrt{\tilde{p}}, \omega_{12}(3) - \frac{1}{\sqrt{\tilde{p}}} + \frac{\kappa_1}{\tilde{p}}) & \text{otherwise,} \end{cases}$$

and, symmetrically for agent 2,

$$x_2(\tilde{p}|\kappa) = \begin{cases} (0, \frac{\omega_{21}(3)}{\tilde{p}} + \kappa_2) & \text{for } \omega_{21}(3) - \sqrt{\tilde{p}} + \tilde{p}\kappa_2 \leqslant 0 \\ (\omega_{21}(3) - \sqrt{\tilde{p}} + \tilde{p}\kappa_2, \frac{1}{\sqrt{\tilde{p}}}) & \text{otherwise.} \end{cases}$$

We note first that, in equilibrium, agent 2 never stores in shock 1 and agent 1 never stores in shock 2. To see this, observe that agent 2 stores in shock 1 only if his or her consumption in good 2 in the subsequent shock 3 is below 0.1. However, $x_2(\tilde{p}|\kappa) \leqslant 0.1$ and $\kappa \geqslant 0$ implies $\omega_{21}(3)/\tilde{p} \leqslant 0.1$, thus the (relative) price of good 2, $\tilde{p}$, must be at least 20. But then agent 1's consumption of good 1 must be at least $\sqrt{20}$, which violates feasibility. Therefore there cannot be an equilibrium where agent 2 stores in shock 1. The situation for shock 2 is completely symmetric—agent 1 will never store in this shock.

We now consider a two-period economy with the initial shock equal to 1 where agent 2 does not store—that is, $\kappa_2 = 0$. If also $\kappa_1 = 0$, then the equilibrium conditions for the second period spot market have a continuum of solutions: any $\tilde{p}$ satisfying $\omega_{12}(3)^{-2} = 1/4 \leqslant \tilde{p} \leqslant \omega_{21}(3)^2 = 4$ is a possible spot market equilibrium. However, we now show that in the two-period economy only $\tilde{p} = 4$ is consistent with agent 1's intertemporal optimization. For $\tilde{p} = 4$, agent 1's consumption at shock 3 is given by $x_1(z' = 3) = (2, 1.5)$. If agent 1's consumption in good 1 drops below 2, he or she will always store positive amounts, and by feasibility it cannot be above 2 without storage. To see that this equilibrium is unique, first observe that there cannot be another equilibrium with identical consumption for agent 1 in good 1. To see that there cannot be an equilibrium with $\kappa_1 > 0$, observe that for $\kappa_1 > 0$ the only possible spot equilibrium would have $x_{11} = 2 + \kappa_1$. However, the Euler equation implies that $\kappa_1 > 0$ is then inconsistent with intertemporal optimality. When the economy starts in shock 2, the situation is completely symmetric, with only one possible equilibrium with $\kappa_1 = \kappa_2 = 0$, $\tilde{p} = \frac{1}{4}$ and agent 1's consumption given by $x_1(z' = 3) = (0.5, 0)$.

Thus, in every competitive equilibrium we have $\kappa_1 = \kappa_2 = 0$ and consumption and prices in shock 3 differ depending on whether the realization of the previous shock was 1 or 2. Therefore, there is no recursive equilibrium.

Clearly, the counterexample relies crucially on the fact that given $\kappa_1 = \kappa_2 = 0$ there are several possible continuation equilibria. As Kubler and Schmedders (2002) point out, the assumption of uniqueness of competitive equilibria for all possible initial conditions ensures the existence of a recursive equilibrium. However, this assumption is highly unreasonable. For models with infinitely lived agents and incomplete financial markets no assumptions are known that guarantee uniqueness. Ever since Kehoe (1985) it has been well known that, even for the static Arrow–Debreu model with production, conditions that guarantee uniqueness of equilibria are too restrictive to have much applicability. Moreover, none of these conditions extend to dynamic stochastic models with incomplete markets. Therefore, we do not try to find conditions that rule out multiple equilibria. Instead, our strategy is to find conditions that ensure, in the presence of multiple equilibria, that there is at least one equilibrium that is recursive in the natural state.

3 Existence

In this section we prove the existence of a recursive equilibrium for the general model presented in Section 2. Section 3.1 shows how to characterize recursive equilibrium via marginal utility functions. Section 3.2 proves existence making direct assumptions on the transition probability for the recursive state. Assumptions on the economic fundamentals which guarantee that these conditions hold are provided in Section 3.3. In Section 3.4 we outline how our results can be extended to allow for financial assets.

3.1 Characterizing recursive equilibria

We now characterize recursive equilibrium via a function that maps the recursive state into marginal utilities of all agents. We show that such a function describes a recursive equilibrium if it is a fixed point of an operator that captures the period-to-period equilibrium conditions.

Since we consider an economy with several commodities we want to allow for the fact that some commodities do not enter the utility functions of agents and some commodities, although their consumption provides utility, are not essential in that an agent might decide to consume zero of that commodity. Nevertheless, we need to assume that there is at least one commodity that is essential in the sense that independently of prices an agent will always consume positive amounts of that commodity. For simplicity we take the consumption space to be $\mathbf{C} = \mathbb{R}_{++} \times \mathbb{R}_{+}^{L-1}$, assuming that utility and marginal utility are well defined even if consumption of goods $2, \ldots, L$ are on the boundary. It is straightforward to amend our proofs and to allow for additional Inada conditions for some, or all, of the commodities $2, \ldots, L$.

We make the following assumption on preferences and endowments:

Assumption 1

1. *Individual endowments in good 1 and aggregate endowments in all other goods are bounded above and bounded below away from zero—that is, there are* $\underline{\omega}, \overline{\omega} \in \mathbb{R}_{++}$ *such that for all shocks* z

$$\underline{\omega} < \omega_{h1}(z) < \overline{\omega} \text{ for all agents } h,$$

$$\underline{\omega} < \frac{1}{H} \sum_{h \in \mathbf{H}} \omega_{hl}(z) < \overline{\omega} \text{ for all goods } l = 2, \ldots, L.$$

2. *The agents' discount factor satisfies* $\delta \in (0, 1)$.

3. *The Bernoulli functions* $u_h : \mathbf{Z} \times \mathbf{C} \to \mathbb{R}$, $h \in \mathbf{H}$, *are measurable in* z, *they are increasing, concave, and continuously differentiable in* x, *they are strictly increasing and strictly concave in* x_1, *and they satisfy a strong Inada condition: for any sequence* $x_1^n \to 0$, *we have* $\sup_{z \in \mathbf{Z}, (x_2, \ldots, x_L) \in \mathbb{R}_+^{L-1}} u_h(z, (x_1^n, x_2, \ldots, x_L)) \to -\infty$. *Utility is bounded above: there exists a* $\bar{u}$ *such that* $u_h(z, x) \leqslant \bar{u}$ *for all* $h \in \mathbf{H}$, $z \in \mathbf{Z}$, $x \in \mathbf{C}$.

The assumption on utility seems strong but it is a direct generalization of the assumptions in Duffie et al. (1994) to an economy with several commodities and continuous shocks. In specific applications, the assumption that individual endowments are strictly positive in the "essential" commodity, good 1, and the assumption that aggregate endowments are positive in all goods can be replaced by alternative assumptions, as we discuss in Section 4.2.

As in Duffie et al. (1994), Assumption 1 implies that there is a $\underline{c} > 0$ such that, independently of prices, an agent will never choose consumption in commodity 1 that is below $\underline{c}$. The reason is that budget feasibility implies that an agent can always consume his or her endowments (the agent cannot sell them on financial markets in advance), and we therefore must have, for any shock z and for any x with $x_1 < \underline{c}$,

$$u_h(z, x) + \frac{\delta \overline{u}}{1 - \delta} < \frac{1}{1 - \delta} \inf_{z \in \mathbf{Z}} u_h(z, \underline{x}),$$

where $\bar{u}$ is the upper bound on Bernoulli utility and $\underline{x}_1 = \underline{\omega}, \underline{x}_l = 0, l = 2, \ldots, L$.
The lower bound on consumption implies an upper bound on marginal utility, which we define by[6]

$$\bar{m} = \max_{h \in \mathbf{H}} \sup_{x \in \mathbb{R}_+^L, x_1 \geqslant \underline{c}/2, z \in \mathbf{Z}} \frac{\partial u_h(z, x)}{\partial x_1}. \tag{1}$$

We make the following assumptions on production possibilities:

Assumption 2 *For each shock* z *the production set* $\mathbf{Y}(z) \subset \mathbb{R}^L$ *is assumed to be closed, convex-valued, to contain* $\mathbb{R}_-^L$, *to exhibit constant returns to scale—that is,* $y \in \mathbf{Y}(z) \Rightarrow \lambda y \in \mathbf{Y}(z)$ *for all* $\lambda \geqslant 0$, *and to satisfy* $\mathbf{Y}(z) \cap -\mathbf{Y}(z) = \{0\}$. *In addition, production is bounded above: There is a* $\bar{\kappa} \in \mathbb{R}_+$ *so that for all* $\kappa \in \mathbf{K}^U$, $h \in \mathbf{H}$, $z \in \mathbf{Z}$, $l \in \mathbf{L}^K$, *and for all* $\alpha \in \mathbb{R}_+^{HJ}$

$$\sum_{h \in \mathbf{H}} \left(A_h^0(z) \alpha_h - \kappa_h - e_h(z) \right) \in \mathbf{Y}(z) \Rightarrow \sup_{z'} \sum_{h \in \mathbf{H}} (f_{hl}(z') + \sum_{j \in \mathbf{J}} a_{hjl}^1(z') \alpha_{hj}) \leqslant \max[\bar{\kappa}, \sum_{h \in \mathbf{H}} \kappa_{hl}].$$

[6] In our results below we will often require that variables are actually bounded away from some lower (or upper) bound $\underline{b}$ (or $\bar{b}$). In order to ensure this, we take a known bound $\underline{a}$ (or $\bar{a}$) in $\mathbb{R}_{++}$ and define $\underline{b} = \underline{a}/2$ ($\bar{b} = 2\bar{a}$).

While the first part of Assumption 2 is standard, the second part is a strong assumption on the interplay of intra- and inter-period production. For each capital good, the economy can never grow above $\bar{\kappa}$ when starting below that limit. The assumption is made for convenience and ensures boundedness of consumption. In specific applications stronger assumptions on the correspondence $\mathbf{Y}(.)$ can lead to a relaxation of the second part of Assumption 2 (see Section 4.2 below).

We define

$$\mathbf{K} = \{\kappa \in \mathbf{K}^U : H\underline{\omega} \leqslant \sum_{h \in \mathbf{H}} \kappa_{hl} \leqslant \bar{\kappa} \text{ for all } l \in \mathbf{L}^K; \;\; \text{if } 1 \in \mathbf{L}^K, \; \kappa_{h1} \geqslant \underline{\omega} \text{ for all } h \in \mathbf{H}\} \tag{2}$$

and take the state space to be $\mathbf{S} = \mathbf{Z} \times \mathbf{K}$ with Borel σ-algebra $\mathcal{S}$. We define $\boldsymbol{\Xi}$ to be the set of storage decisions across agents, α, that ensure that next period's endogenous state lies in $\mathbf{K}$:

$$\boldsymbol{\Xi} = \{\alpha \in \mathbb{R}_+^{HJ} : \left(f_h(z') + A_h^1(z')\alpha_h\right)_{h \in \mathbf{H}} \in \mathbf{K} \text{ for all } z' \in \mathbf{Z}\}. \tag{3}$$

The following proposition gives a characterization of recursive equilibria that is at the heart of our existence proof below.

Proposition 1 *Suppose Assumptions 1 and 2 hold. Then a recursive equilibrium exists if there are bounded functions* $M : \mathbf{S} \to \mathbb{R}_+^{HL}$ *such that for each* $s = (z, \kappa) \in \mathbf{S}$ *there exist prices* $\bar{p} \in \Delta^{L-1}$, $\bar{p}_1 > 0$, *production plans* $\bar{y} \in \mathbf{Y}(z)$, *and choices* $\{(\bar{x}_h, \bar{\alpha}_h)_{h \in \mathbf{H}}$ *with* $\bar{\alpha} \in \boldsymbol{\Xi}$ *such that for each* $h \in \mathbf{H}$ *,*

$$M_{h1}(s) = \frac{\partial u_h(z, \bar{x}_h)}{\partial x_1}, \quad M_{hl}(s) = M_{h1}(s)\frac{\bar{p}_l}{\bar{p}_1}, \; l = 2, \ldots, L$$

and

$$\begin{aligned}(\bar{x}_h, \bar{\alpha}_h) \;\; \in \;\; \arg &\max_{x_h \in \mathbf{C}, \alpha_h \in \mathbb{R}_+^J} u_h(z, x_h) + \delta \mathbb{E}_s\left[M_h(s')A_h^1(z')\alpha_h\right] \;\; \textit{s.t.} \\ &-\bar{p} \cdot (x_h + A_h^0(z)\alpha_h - \kappa_h - e_h(z)) \geqslant 0,\end{aligned} \tag{4}$$

where

$$s' = \left(z', \left(A_h^1(z')\bar{\alpha}_h + f_h(z')\right)_{h \in \mathbf{H}}\right),$$

production plans are optimal,

$$\bar{y} \in \arg \max_{y \in \mathbf{Y}(z)} \bar{p} \cdot y,$$

and markets clear,

$$\sum_{h \in \mathbf{H}} (\bar{x}_h + A_h^0(z)\bar{\alpha}_h - e_h(z) - \kappa_h) = \bar{y}.$$

The key idea of this proposition is that the first order conditions of (4) are identical to the agents' intertemporal Euler equations. The proof proceeds by showing that these Euler equations are necessary and sufficient for optimal intertemporal choices. The alternative characterization of a recursive equilibrium in terms of M-functions provided in Proposition 1 is useful because it allows us to show existence through a fixed-point argument in the space of these marginal utility functions.

This strategy of proof is possible as we can (under suitable additional assumptions) show that for any given measurable and bounded function $M(.)$ there exist prices and choices satisfying the conditions in Proposition 1. This is formalized in Lemma 2 below.

3.2 Existence under assumptions on the transition

Using the characterization of recursive equilibrium given in Proposition 1, we now prove its existence by making direct assumptions on the function that maps the current recursive state and current actions into the probability distribution of next period's recursive state. In Section 3.3 we provide concrete conditions on the exogenous shocks that are sufficient to ensure that these assumptions hold.

Assuming that the probability distribution of the next period's state varies continuously with current actions, we will show that the operator defined by the equilibrium conditions is a non-empty correspondence on the space of marginal utility functions. By the Fan–Glicksberg fixed-point theorem this implies the existence of a fixed point of the convex hull of this correspondence. Making an additional assumption that ensures the presence of "noise" as in Duggan (2012)—the actual assumption we make is from He and Sun (2017)—we can prove the existence of a recursive equilibrium. Note that, in general, continuation equilibria will not be unique and our assumptions imply nothing about their uniqueness. However, Assumptions 3.1 and 3.2 below ensure that there exists a measurable selection of continuation equilibria whose conditional expectation is a continuous function of today's choices. Furthermore, Assumption 3.3 ensures that any measurable selection of the convex hull of all continuation equilibria is itself a continuation equilibrium. While our assumptions do not rule out multiple continuation equilibria, they guarantee the existence of a recursive equilibrium.

To state the assumptions formally, first note that the exogenous transition probability $\mathbb{P}$ implies, given choices $\alpha \in \Xi$, a transition probability $\mathbb{Q}(.|s, \alpha)$ on $\mathcal{S}$: given α across all agents, and next period's shock z', the next period's endogenous state is given by

$$\left(f_h(z') + A_h^1(z')\alpha_h\right)_{h \in \mathbf{H}}.$$

To prove the existence of a recursive equilibrium we first make additional assumptions directly on $\mathbb{Q}$. To state them we need the following definition from He and Sun (2017): Given a measure space $(\mathbf{S}, \mathcal{S})$ with an atomless probability measure λ and a sub-σ-algebra $\mathcal{G}$, let $\mathcal{G}^B$ and $\mathcal{S}^B$ be defined as $\{B \cap B' : B' \in \mathcal{G}\}$ and $\{B \cap B' : B' \in \mathcal{S}\}$, for any non-negligible set $B \in \mathcal{S}$. A set $B \in \mathcal{S}$ is said to be a *$\mathcal{G}$-atom* if $\lambda(B) > 0$ and given any $B_0 \in \mathcal{S}^B$, there exists a $B_1 \in \mathcal{G}^B$ such that $\lambda(B_0 \triangle B_1) = 0$.

The following assumptions are from He and Sun (2017)[7]—in Section 3.3 we give assumptions

[7] Assumptions 3.1 and 3.2 correspond to the assumptions made by He and Sun (2017) on the transition probability representing the law of motion of the states. Assumption 3.3 corresponds to their crucial sufficient condition for existence, called the "coarser transition kernel".

on fundamentals that imply Assumption 3 and thereby ensure existence.

Assumption 3

1. *For any sequence* $\alpha^n \in \Xi$ *with* $\alpha^n \to \alpha^0 \in \Xi$,

$$\sup_{B \in \mathcal{S}} |\mathbb{Q}(B|s, \alpha^n) - \mathbb{Q}(B|s, \alpha^0)| \to 0.$$

2. *For all* (s, α), $\mathbb{Q}(.|s, \alpha)$ *is absolutely continuous with respect to the probability measure* λ *on* $(\mathbf{S}, \mathcal{S})$ *with Radon–Nikodym derivative* $q(.|s, \alpha)$.

3. *There is a sub* σ*-algebra* $\mathcal{G}$ *of* $\mathcal{S}$ *such that* $\mathcal{S}$ *has no* $\mathcal{G}$ *atom and* $q(.|s, \alpha)$ *and* $A^1(.)$ *are* $\mathcal{G}$*-measurable for all* $s = (z, \kappa)$ *and all* $\alpha \in \Xi$.

The first existence result of this paper is as follows:

Proposition 2 *Under Assumptions 1–3 a recursive equilibrium exists.*

To prove the result, let $L^m_\infty(\mathbf{S}, \mathcal{S}, \lambda)$ be the space of essentially bounded and measurable (equivalence classes of) functions from $\mathbf{S}$ to $\mathbb{R}^m$ with $m = HL$. Following Nowak and Raghavan (1992) and Duggan (2012), we endow L^m_∞ with the weak* topology $\sigma(L^m_\infty, L^m_1)$. For any $b > 0$ the set of measurable functions that are λ-essentially bounded above by b and below by 0 is then a non-empty, convex, and weak* compact subset of a locally convex, Hausdorff topological vector space. We denote this set by $\mathbf{M}^b \subset L^m_\infty$. Since $\mathbf{S}$ is a separable metric space, L^m_1 is separable, and consequently $\mathbf{M}^b$ is metrizable in the weak* topology. We define $L^+_\infty \subset L^m_\infty$ to be the set of functions in L^m_∞ that are essentially bounded below by zero. Given any $\bar{M} = (\bar{M}^1, \ldots, \bar{M}^H) \in L^+_\infty$, we define

$$E^{\bar{M}}_h(s, x_h, \alpha_h, \alpha^*) = u_h(z, x_h) + \delta \mathbb{E}_s \left[\bar{M}^h(s') \cdot A^1_h(z') \alpha_h \right] \tag{5}$$

with

$$s' = \left(z', \left(f_h(z') + A^1_h(z') \alpha^*_h\right)_{h \in \mathbf{H}}\right).$$

In the definition of $E^{\bar{M}}_h$, the $\alpha_h \in \mathbb{R}^J_+$ stands for the choice of agent h, while $\alpha^* \in \mathbb{R}^{HJ}_+$ is taken by individuals as given—in particular its influence on the state transition. Lemma 1 states the properties of the function $E^{\bar{M}}_h$ that we need in Lemma 2.

Lemma 1 *Given any* $\bar{M} \in L^+_\infty$ *and* $h \in \mathbf{H}$, *the function* $E^{\bar{M}}_h(., x_h, \alpha_h, \alpha^*)$ *is measurable in* s. *For given* s, *the function is jointly continuous in* x_h, α_h, α^* *and* $\bar{M}$.

The next lemma is the key result in this subsection and it guarantees the existence of a policy in the current period that satisfies the equilibrium conditions, given arbitrary, measurable, and bounded marginal utilities in the subsequent period.[8] The key idea is that Lemma 1 implies that the agents'

[8]In our setup this result plays the same role as the result that there always exists a mixed strategy Nash equilibrium for the stage game in the stochastic game setup.

objective functions are continuous and a standard fixed-point argument can be employed to show the existence of market clearing prices in the current period for any (bounded and measurable) continuation marginal utility in the subsequent period.

Lemma 2 *For each $b > 0$ there is an $\epsilon > 0$ such that for any $\bar{M} \in \mathbf{M}^b$ and all $s = (z, \kappa) \in \mathbf{S}$ there exist $\bar{x} \in \mathbb{R}_+^{HL}$, $\bar{\alpha} \in \mathbf{\Xi}$, $\bar{y} \in \mathbf{Y}(z)$, and $\bar{p} \in \Delta^{L-1}$ with $\bar{p}_1 \geqslant \epsilon$ such that*

$$\sum_{h \in \mathbf{H}} (\bar{x}_h + A_h^0(z)\bar{\alpha}_h - e_h(z) - \kappa_h) = \bar{y}, \tag{6}$$

for each agent h

$$\begin{aligned} (\bar{x}_h, \bar{\alpha}_h) \in \arg \max_{x_h \in \mathbf{C}, \alpha_h \in \mathbb{R}_+^J} & E_h^{\bar{M}}(s, x_h, \alpha_h, \bar{\alpha}) \text{ s.t.} \\ -\bar{p} \cdot (x_h - e_h(z) - \kappa_h + A_h^0(z)\alpha_h) & \geqslant 0, \end{aligned} \tag{7}$$

and

$$\bar{y} \in \arg \max_{y \in \mathbf{Y}(z)} \bar{p} \cdot y. \tag{8}$$

For a given $\bar{M}$, we define the (consumption) correspondence $s \rightrightarrows \mathbf{N}_{\bar{M}}(s)$ to contain all $((x_h)_{h \in \mathbf{H}}, p)$ such that there exist $(\alpha_h)_{h \in \mathbf{H}} \in \mathbf{\Xi}$ and $y \in \mathbf{Y}(z)$ that satisfy Equations (6), (7), and (8). We define the associated (marginal utility) correspondence $s \rightrightarrows \mathbf{P}_{\bar{M}}(s)$ by

$$\mathbf{P}_{\bar{M}}(s) = \left\{ \left(\frac{\partial u_h(z, x_h)}{\partial x_{h1}}, \frac{p_2}{p_1} \frac{\partial u_h(z, x_h)}{\partial x_{h1}}, \ldots, \frac{p_L}{p_1} \frac{\partial u_h(z, x_h)}{\partial x_{h1}} \right)_{h \in \mathbf{H}} : (x, p) \in \mathbf{N}_{\bar{M}}(s) \right\},$$

and $\mathbf{P}_{\bar{M}}^{co}$ by requiring

$$\mathbf{P}_{\bar{M}}^{co}(s) = \operatorname{conv}(\mathbf{P}_{\bar{M}}(s)) \text{ for all } s \in \mathbf{S},$$

where $\operatorname{conv}(A)$ denotes the convex hull of a set A. Let $\mathbf{R}(\bar{M})$ be the set of (equivalence classes of) measurable selections of $\mathbf{P}_{\bar{M}}$, and $\mathbf{R}^{co}(\bar{M})$ the set of measurable selections of $\mathbf{P}_{\bar{M}}^{co}$. Note that for any $\mathbf{M} \subset \mathbf{M}^b$ this defines a correspondence $\mathbf{R}^{co} : \mathbf{M} \rightrightarrows L_\infty^+$. In the following, we first establish that for convex and closed domains $\mathbf{M}$ this correspondence has a closed graph and non-empty, convex values. Then we go on to show that the set $\mathbf{M}$ can be chosen to ensure that $\mathbf{R}^{co}$ maps into $\mathbf{M}$ and that a fixed point of this map describes a recursive equilibrium as in Proposition 1. The following lemma is an important but standard technical result (see, e.g., Nowak and Raghavan (1992)).

Lemma 3 *For each $\bar{M} \in L_\infty^+$, the correspondence $\mathbf{P}_{\bar{M}}(s)$ is weakly measurable and compact valued, and for any $b > 0$ and any weak* closed and convex $\mathbf{M} \subset \mathbf{M}^b$ the correspondence $\mathbf{R}^{co} : \mathbf{M} \rightrightarrows L_\infty^m$ is non-empty, convex, weak* closed valued, and has a weak* closed graph.*

As explained in the introduction, our existence proof relies on the Fan–Glicksberg fixed-point theorem, which will guarantee the existence of a fixed point of $\mathbf{R}^{co}$. In order to deduce from that the existence of a recursive equilibrium we follow a similar approach as He and Sun (2017).

Lemma 4 *Let $\mathbf{F} : \mathbf{S} \rightrightarrows \mathbb{R}^{HL}$ be an integrably bounded and closed valued correspondence and define $\mathbf{F}^{co}(s) = conv(\mathbf{F}(s))$ for all $s \in \mathbf{S}$. Let $M(s) = (M_1^h(s), ..., M_L^h(s))_{h=1}^H$ be a measurable selection of $\mathbf{F}^{co}$. Then there exists an $\hat{M}$ that is a measurable selection of $\mathbf{F}$ such that for all $h \in \mathbf{H}$, $s \in \mathbf{S}$, $\alpha \in \Xi$*

$$\int_{\mathbf{S}} M_h(s') A_h^1(z') d\mathbb{Q}(s'|s, \alpha) = \int_{\mathbf{S}} \hat{M}_h(s') A_h^1(z') d\mathbb{Q}(s'|s, \alpha).$$

For arbitrary $\mathbf{M}^b$ the correspondence $\mathbf{R}^{co} : \mathbf{M}^b \rightrightarrows L_\infty^m$ does not necessarily map into $\mathbf{M}^b$. The final lemma of this section establishes the existence of a suitable subset of L_∞^+, which can be used for the fixed-point argument.

Lemma 5 *There exists a convex and weak* compact set $\mathbf{M}^* \subset L_\infty^+$ such that $\mathbf{R}^{co}(\bar{M}) \subset \mathbf{M}^*$ for all $\bar{M} \in \mathbf{M}^*$.*

To complete the proof of the existence of a recursive equilibrium—that is to say, the proof of Proposition 2, recall the statement of the Fan–Glicksberg theorem (see, e.g., Aliprantis and Border (2006, Theorem 17.55)). Suppose $\mathbf{M}$ is a non-empty compact convex subset of a locally convex Hausdorff topological vector space; then a correspondence $\mathbf{M} \rightrightarrows \mathbf{M}$ has a fixed point if it has closed graph and non-empty convex values. For $\mathbf{M}^*$ as in Lemma 5, by Lemma 3 and Fan–Glicksberg fixed-point theorem there exists a $\hat{M} \in \mathbf{M}^*$ such that $\hat{M} \in \mathbf{R}^{co}(\hat{M})$. By Lemma 4 it is then clear that for all s, $\mathbf{P}_{\hat{M}}(s) = \mathbf{P}_{M^*}(s)$ and M^* must be a $\mathcal{S}$-measurable selection of $\mathbf{P}_{M^*}(s)$. Therefore there exists a bounded function M^* that satisfies the conditions of Proposition 1 and a recursive equilibrium exists.

3.3 The existence theorem

So far, we have shown the existence of a recursive equilibrium under Assumptions 1–3. However, Assumption 3 is not a direct assumption on the fundamentals of the economy, but rather on how the transition probability for exogenous and endogenous states varies with choices. We now provide concrete assumptions on the stochastic process of exogenous shocks—assumptions that guarantee Assumption 3 and thus the existence of a recursive equilibrium.

In particular, we now assume that the space of exogenous shocks can be decomposed into three complete, separable metric spaces, $\mathbf{Z} = \mathbf{Z}_0 \times \mathbf{Z}_1 \times \mathbf{Z}_2$ with Borel σ-algebra $\mathcal{Z} = \mathcal{Z}_0 \otimes \mathcal{Z}_1 \otimes \mathcal{Z}_2$, and the shock is given by $z = (z_0, z_1, z_2)$. Moreover, for each $i = 0, 1, 2$ there is a measure μ_{z_i} on $\mathbf{Z}_i$ and there are conditional densities $r_{z_0}(z_0'|z, z_1')$, $r_{z_1}(z_1'|z)$, and $r_{z_2}(z_2'|z, z_0', z_1')$ such that for any $B \in \mathcal{Z}$ we have

$$\mathbb{P}(B|z) = \int_{\mathbf{Z}_1} \int_{\mathbf{Z}_0} \int_{\mathbf{Z}_2} \mathbb{1}_B(z') r_{z_2}(z_2'|z, z_0', z_1') r_{z_0}(z_0'|z, z_1') r_{z_1}(z_1'|z) d\mu_{z_2}(z_2') d\mu_{z_0}(z_0') d\mu_{z_1}(z_1').$$

To ensure the continuity of the state transition in Assumption 3.1, we assume that the shock z_0 is purely transitory, has a continuous density, and only affects agents' f-endowments. Moreover, given z_1 and z_2, there is a diffeomorphism from $\mathbf{Z}_0$ to a subset of $\mathbf{K}$. More precisely, we make the following assumptions:

ASSUMPTION 4

1. *z_0 is purely transitory—that is, for all $z_0, \hat{z}_0 \in \mathbf{Z}_0$ and all $(z_1, z_2) \in \mathbf{Z}_1 \times \mathbf{Z}_2$,*

$$\mathbb{P}(.|z_0, z_1, z_2) = \mathbb{P}(.|\hat{z}_0, z_1, z_2).$$

2. *$\mathbf{Z}_0$ is a subset of an Euclidean space, μ_{z_0} is Lebesgue, and the density $r_{z_0}(.|z, z_1')$ is continuous for almost all (z, z_1').*

3. *For all $(z_1, z_2) \in Z_1 \times Z_2$, $(f_h(., z_1, z_2))_{h\in\mathbf{H}}$ is a C^1-diffeomorphism from $\mathbf{Z}_0$ to a subset of $\mathbf{K}$ with a non-empty interior. All other fundamentals are independent of z_0—that is, for all h, we can write $e_h(z) = e_h(z_1, z_2)$, $A_h^0(z) = A_h^0(z_1, z_2)$, $A_h^1(z) = A_h^1(z_1, z_2)$, $u_h(z, .) = u_h((z_1, z_2), .)$, $\mathbf{Y}(z) = \mathbf{Y}(z_1, z_2)$.*

Assumption 4.3. can be slightly relaxed in that we can allow $A_h^1(z)$ to depend on z_0 if we assume that for all $\alpha \geqslant 0$, $\left(f_h(., z_1', z_2') + A_h^1(., z_1', z_2')\alpha\right)_{h\in\mathbf{H}}$ is a diffeomorphism from $\mathbf{Z}_0$ to a subset of $\mathbb{R}^{HL}$. For simplicity we take $A_h^1(.)$ to be independent of z_0.
To ensure that the z_2 shock gives us convexity in the conditional expectation operator we make the following assumption:

ASSUMPTION 5 *Conditionally on next period's z_1', the shock z_2' is independent of both z_0' and the current shock z. Conditionally on z_1', the measure $\mu_{z_2}(.|z_1')$ is absolutely continuous with respect to some atomless probability measure on $\mathbf{Z}_2$ so that we can write the density as $r_{z_2}(z_2'|z, z_0', z_1') = r_{z_2}(z_2'|z_1')$. Moreover, for each agent h, $A_h^1(z)$ and $f_h(z)$ do not depend on z_2.*

This construction was first used in Duggan (2012). It is clear that this is a strict generalization of a "sunspot". The shock z_2 can affect fundamentals $(e_h, u_h)_{h\in\mathbf{H}}$ and $\mathbf{Y}$ in arbitrary ways.

The following is the main result of the paper.

THEOREM 1 *Under Assumptions 1, 2, 4, and 5 there exists a recursive equilibrium.*

To prove the theorem we show that Assumptions 4 and 5 imply Assumption 3 if state transitions and the state space are reformulated appropriately. It is easy to notice that since the shock z_0 is purely transitory and does not affect any fundamentals except (f_h), the realization of this shock is reflected in the value of the endogenous state κ and except for the value of κ it is irrelevant for current endogenous variables and the future evolution of the economy. Therefore, departing slightly from our previous notation, we take $\mathbf{S} = \mathbf{Z}_1 \times \mathbf{Z}_2 \times \mathbf{K}$ with Borel σ-algebra $\mathcal{S}$. Furthermore, we write $\underline{\mathbf{S}} = \mathbf{Z}_1 \times \mathbf{K}$ for the space that includes only the z_1-shock component and the holdings in capital goods; we denote the Borel σ-algebra on $\underline{\mathbf{S}}$ by $\underline{\mathcal{S}}$. For each $B \in \mathcal{S}$, take

$$\mathbb{Q}(B|s,\alpha) = \mathbb{P}\left(\{z' \in \mathbf{Z} : \left((z_1', z_2'), (f_h(z') + A_h^1(z')\alpha_h)_{h\in\mathbf{H}}\right) \in B\}|z\right).$$

The following lemma establishes that Assumptions 4 and 5 provide sufficient conditions for Assumption 3 and hence for the existence of a recursive equilibrium.

Lemma 6 *Under Assumptions 4 and 5 ,* $\mathbb{Q}(.|s,\alpha)$ *satisfies Assumption 3.*

The proof of Theorem 1 now follows directly from the argument above—that is to say, the result follows directly from Proposition 2.

3.4 Financial markets

So far, we have considered the case without trade in one-period financial assets. We now briefly outline how financial markets can be incorporated into our framework. More precisely, we assume that agents can trade in financial assets, in addition to undertaking intertemporal storage. There are D one-period securities, $d = 1, ..., D$, in zero net supply, each being characterized by its payoff $b_d : \mathbf{Z} \to \mathbb{R}_+^L$, which is a bounded and measurable function of the shock. At each z^t, securities are traded at prices $q(z^t)$; we denote an agent's portfolio by $\theta_h(z^t) \in \mathbb{R}^D$.

In order to establish the existence of a recursive equilibrium we need to restrict agents' portfolio choices. Let $\mathbf{K}$ be defined as in (2) above and $\mathbf{\Xi}$ as in (3). Each agent h faces a constraint on trades in asset markets and storage decisions (α, θ), given by a convex and closed set $\mathbf{\Theta}_h \subset \mathbb{R}_+^J \times \mathbb{R}^D$, which satisfies that whenever $\alpha \in \mathbf{\Xi}$ and $(\alpha, \theta) \in \mathbf{\Theta}_h$ then

$$A_h^1(z')\alpha + \sum_{d=1}^{D} \theta_d b_d(z') \geqslant 0 \text{ for all } z' \in \mathbf{Z}.$$

Without loss of generality we assume that trade is possible in all financial securities—that is, for each d there is an agent h and an $\alpha \in \mathbf{\Xi}$ so that for some $\theta_d < 0$, $(\alpha, \theta) \in \mathbf{\Theta}_h$. Note that collateral constraints of the form

$$A_h^1(z')\alpha + \sum_{d=1}^{D} \min(\theta_d, 0) b_d(z') \geqslant 0 \text{ for all } z' \in \mathbf{Z} \tag{9}$$

are one example of constraints that satisfy our assumption. However, this is a somewhat nonstandard formulation of a collateral constraint since agents cannot borrow against the value of their future production—they need to borrow against future production directly.

As before, the endogenous state space is given by $\mathbf{K}$. A recursive equilibrium is given by maps from the state $s \in \mathbf{S} = \mathbf{Z} \times \mathbf{K}$ to prices of commodities and financial securities and to consumption, investment, and portfolio choices across all agents. The analogous result to above is now as follows:

A recursive equilibrium exists if there are functions $M : \mathbf{S} \to \mathbb{R}_+^{HL}$ such that for each $s \in \mathbf{S}$ there exist prices $(\bar{p}, \bar{q}) \in \Delta^{L+D-1}$, a production plan $\bar{y} \in \mathbf{Y}(z)$ for each agent h, optimal actions $(\bar{x}_h, \bar{\alpha}_h, \bar{\theta}_h)$ with

$$M_{h1}(s) = \frac{\partial u_h(z, x_h)}{\partial x_1}, \quad M_{hl}(s) = M_{h1}(s)\frac{\bar{p}_l}{\bar{p}_1},\ l = 2, \ldots, L$$

such that

$$\begin{aligned}(\bar{x}_h, \bar{\alpha}_h, \bar{\theta}_h) \quad \in \quad & \arg \max_{x \in \mathbf{C}, (\alpha, \theta) \in \mathbf{\Theta}_h} \\ & u_h(z, x) + \delta \mathbb{E}_s \left[M_h(s') \cdot \left(\sum_j a^1_{hj}(z')\alpha_j + \sum_d b_d(z')\theta_d \right) \right] \text{ s.t.} \\ & -\bar{q} \cdot \theta - \bar{p} \cdot (x + A^0_h(z)\alpha - \kappa_h - e_h(z)) \geqslant 0,\end{aligned}$$

where

$$s' = \left(z', \left(A^1_h(z')\bar{\alpha}_h + \sum_d \bar{\theta}_{hd} b_d(z') + f_h(z') \right)_{h \in \mathbf{H}} \right),$$

production plans are optimal,

$$\bar{y} \in \arg \max_{y \in \mathbf{Y}(z)} \bar{p} \cdot y,$$

and markets clear,

$$\sum_{h \in \mathbf{H}} (\bar{x}_h + A^0_h(z)\bar{\alpha}_h - e_h(z) - \kappa_h) = \bar{y},$$

and

$$\sum_{h \in \mathbf{H}} \bar{\theta}_h = 0.$$

The proof is similar to the proof of Proposition 1.

Assumptions 4.3 and 5 now need to be extended: We assume in addition that for each asset d, $b_d(z)$ is only a function of z_1. The definition of the transition probability $\mathbb{Q}$ now reads as

$$\mathbb{Q}(B|s, \alpha, \theta) = \mathbb{P}\left(\{z' \in \mathbf{Z} : [z'_1, z'_2, (f_h(z') + A^1_h(z')\alpha_h + \sum_d \theta_{hd} b_d(z'))_{h \in \mathbf{H}}] \in B\} | z \right).$$

With the additional assumptions, it is easy to see that Lemma 6 holds as stated. The proofs of Lemmas 1, 3, and 4 are almost identical to those for the case without financial securities. To prove the analogue of Lemma 2, one can bound the set of admissible portfolios, and proceed as in the proof of that lemma. To prove the analogue of Lemma 5 it is necessary to make more precise the constraints subsumed in the set $\mathbf{\Theta}$—it is easy to see that the proof of the lemma goes through for the case of collateral constraints (9).

4 Applications

To illustrate the usefulness of the results obtained in our general model we consider heterogeneous agent versions of the Lucas (1978) asset pricing model and the Brock and Mirman (1972) stochastic growth model. We explain that these models can be analyzed as special cases of our general setup and provide conditions that ensure the existence of a recursive equilibrium. For the Lucas model our sufficient conditions for existence are much stronger than in Duffie et al. (1994): We assume that the Lucas tree holdings are subject to displacement risk[9] and that there is an atomless noise shock that may effect endowments or preferences. For the neoclassical growth model we assume that shocks to labor-endowments have an i.i.d. component with continuous density and that there is an atomless noise shock that may affect preferences or convey news about the probability of future shocks. While our model has similarities to the models in Krusell and Smith (1998) and in Miao (2006) it is important to note that it differs in two crucial aspects. We assume that there are finitely many types of agents and we consider a structure of stochastic shocks that is considerably more complicated than in these paper. In the conclusion of this paper we explain why our method of proof cannot be used to obtain existence of a recursive equilibrium without such strong assumptions.

4.1 A Lucas asset pricing model with displacement risk

In the heterogeneous agent version of the Lucas (1978) asset pricing model that is examined in Duffie et al. (1994) there are J Lucas trees available for trade, $j \in \mathbf{J} = \{1, \ldots, J\}$. These are long-lived assets in unit net supply that pay exogenous positive dividends in terms of the single consumption good. Agents can trade in these trees but are not allowed to hold short positions and there are no other financial securities available for trade. In our setup this amounts to assuming that there are $2J + 1$ commodities, the first being the consumption good, the next J representing the *old* trees, and the last J the *new* trees; there are J linear, intraperiod production technologies each using one particular commodity $j = 2, ..., J + 1$ as input, generating the same amount of commodity $j = J + 2, ..., 2J + 1$ as output, and also producing some amount of commodity 1. Finally, in intertemporal production each agent can store each commodity $j = J + 2, ..., 2J + 1$, which then yields the same amount of commodity $j = 2, ..., J + 1$ in the next period. Agents only derive utility from consumption of commodity 1 and have positive state-contingent individual endowments only in this commodity—except for $t = 0$ when agents have initial endowments in commodities $j = 2, ..., J + 1$ that add up to 1. It is easy to see that a sequential competitive equilibrium for this version of our model will have the same consumption allocation as a sequential equilibrium in the heterogeneous agent Lucas model. This exact model, however, does not satisfy the assumptions needed for Theorem 1, as endowments in the Lucas trees are assumed to be zero. In contrast, Assumption 4 demands that these endowments are “sufficiently stochastic” to make the state transition norm-continuous. In the displacement risk model that we now present, endowments in Lucas trees are stochastic because “new ideas replace old ideas” and thus part of the old Lucas tree

[9]Modeling the redistributive effects of innovation as “displacement risk” has, in recent years, become popular in the asset pricing literature (see Garleanu et al. 2012a, Garleanu et al. 2012b). Introducing an assumption in the spirit of this literature naturally implies a norm-continuous transition as required for our existence result.

holdings are lost and replaced by new holdings of (potentially) other agents. These new holdings are modeled as endowments in the Lucas trees. Thus, compared to the above description of the Lucas tree model, we now assume that the intertemporal storage technologies for the Lucas trees are risky and that endowments in the Lucas trees are stochastic.

The general model description from Section 2.1 still applies, yet substantially simplifies. Denoting endowments in the consumption good and the Lucas trees by $e_h(z_t) \in \mathbb{R}_{++}$ and $f_h(z_t) \in \mathbb{R}^J_+$, respectively, dividends of the Lucas trees by $d(z) \in \mathbb{R}^J_+$, holdings (i.e., storage choices) in the tree by $\phi_h(z^t) \in \mathbb{R}^J_+$, and the fractions of Lucas trees that are displaced by $D(z_t) = \sum_{h \in \mathbf{H}} f_h(z_t)$, with $0 < D_j(z_t) < 1$ for all $j \in \mathbf{J}$, we can define a *sequential competitive equilibrium* as follows: given initial conditions $(\phi_h(z^{-1}))_{h \in \mathbf{H}} \in \left(\Delta^{H-1}\right)^J$ a sequential competitive equilibrium is a process of $\mathcal{F}_t$-adapted prices and choices,

$$\left(p_t, (x_{h,t}, \phi_{h,t})_{h \in \mathbf{H}}\right)_{t=0}^{\infty},$$

such that markets clear and agents optimize—that is, (A) and (B) hold.

(A) Market clearing:

$$\sum_{h \in \mathbf{H}} \left(x_h(z^t) - e_h(z^t)\right) - d(z^t) \leqslant 0, \sum_{h \in \mathbf{H}} \phi_h(z^t) = 1, \quad \text{for all } z^t.$$

(B) Each agent $h = 1, \dots, H$ maximizes utility:

$$(x_h, \phi_h) \in \arg \max_{(x,\phi) \geqslant 0} U_h(x)$$

$$\text{s.t. } p_1(z^t)\left(x(z^t) - e_h(z_t) - d(z_t)\phi(z^t)\right) + p_2(z^t)\left(\phi(z^t) - \kappa_h(z^t)\right) \leqslant 0, \quad \text{for all } z^t,$$

where we define the Lucas tree holdings at the beginning of the period by

$$\kappa_h(z^t) = (1 - D(z_t))\phi_h(z^{t-1}) + f_h(z_t).$$

Note that we have replaced the profit maximization condition from the general model with market clearing conditions for the Lucas trees as intraperiod production is trivial—there is only one use for each capital good. The endogenous part of the state is now the Lucas tree holdings of all agents at the beginning of the period (after displacement), κ, that is

$$\mathbf{K} = \{\kappa \in \mathbf{R}^{HJ} : \sum_{h \in \mathbf{H}} \kappa_{hj} = 1, 0 \leqslant \kappa_{hj} \leqslant 1 \text{ for all } j \in \mathbf{J} \text{ and all } h \in \mathbf{H}\} = \left(\Delta^{H-1}\right)^J. \tag{10}$$

With this slightly different choice for the endogenous state space, $\mathbf{K}$, and the according change of the set of admissible portfolio holdings $\mathbf{\Xi}$, the proof from above goes through with only little change. Assumption 1 remains unchanged, while the analogue of Assumption 2 is now much more specific; it reads as follows.

Assumption 6

1. *Dividends of the Lucas trees are bounded above and below: There is a* $\overline{\omega} \in \mathbb{R}_{++}$ *such that* $0 < d_j(z) < \overline{\omega}$ *for all* $j \in \mathbf{J},\ z \in \mathbf{Z}$.

2. *Displacement is equal to the endowments in the Lucas tree and strictly between zero and one:* $0 < D_j(z_t) = \sum_{h\in\mathbf{H}} f_{h,j}(z_t) < 1$ *for all* $j \in \mathbf{J},\ z \in \mathbf{Z}$.

To state the analogues to Assumptions 4 and 5, we assume the same decomposition of the exogenous shocks, $\mathbf{Z} = \mathbf{Z}_0 \times \mathbf{Z}_1 \times \mathbf{Z}_2$ with Borel σ-algebra $\mathcal{Z} = \mathcal{Z}_0 \otimes \mathcal{Z}_1 \otimes \mathcal{Z}_2$, such that we have for any $B \in \mathcal{Z}$

$$\mathbb{P}(B|z) = \int_{\mathbf{Z}_1}\int_{\mathbf{Z}_0}\int_{\mathbf{Z}_2} \mathbb{1}_E(z')r_{z_2}(z_2'|z, z_0', z_1')r_{z_0}(z_0'|z, z_1')r_{z_1}(z_1'|z)d\mu_{z_2}(z_2')d\mu_{z_0}(z_0')d\mu_{z_1}(z_1').$$

We take Assumptions 4.1 and 4.2 from above and replace Assumption 4.3 and Assumption 5 by the following assumption.

Assumption 7

1. *For each agent* h*, and all* $(z_1, z_2) \in Z_1 \times Z_2$*,* $(f_{hj}(., z_1, z_2)/D_j(., z_1, z_2))_{h\in\mathbf{H}, j\in\mathbf{J}}$ *is a* C^1*-diffeomorphism from* $\mathbf{Z}_0$ *to a subset of* $\mathbf{K}$ *with a non-empty interior. All fundamentals except* $f(z)$ *and* $D(z)$ *are independent of* z_0*.*

2. *Conditionally on next period's* z_1' *the shock* z_2' *is independent of both* z_0' *and the current shock* z*. Conditionally on* z_1' *the measure* $\mu_{z_2}(.|z_1')$ *is absolutely continuous with respect to some atomless probability measure on* $\mathbf{Z}_2$*, so that we can write the density as* $r_{z_2}(z_2'|z, z_0', z_1') = r_{z_2}(z_2'|z_1')$*. Moreover,* $f(z)$ *and* $D(z)$ *do not depend on* z_2*.*

Note that Assumption 7.1 does not rule out that $D(z)$ is independent of z_0, as it is in Corollary 1 below. All in all, the following theorem follows directly from Proposition 2 and Theorem 1 above.

Proposition 3 *Under Assumptions 1, 6, 4.1, 4.2, and 7 there exists a recursive equilibrium.*

For illustration purposes we now provide a concrete specification for the stochastic structure of the economic fundamentals that satisfies Assumptions 6, 4.1, 4.2, and 7.

Corollary 1 *Suppose displacement is strictly between zero and one,* $0 < D(z_t) < 1$*, and endowments in the Lucas tree are given by* $f_h(z_t) = D(z_t) \cdot \tilde{f}_h(z_t)$*, where* $\tilde{f}_h(z_t) \in (\Delta^{H-1})^J$ *is i.i.d. and has a continuous density. Suppose further that* $0 < d_j(z) < \overline{\omega}$ *for all* $j \in \mathbf{J},\ z \in \mathbf{Z}$ *and that aggregate output,* $\sum_{j\in\mathbf{J}} d_j(z_t) + \sum_{h\in\mathbf{H}} e_h(z_t)$*, can be written as the sum of an i.i.d. component that has continuous density over a compact subset of* $\mathbb{R}_{++}$ *and of a Markovian component. Then assumptions 6, 4.1, 4.2, and 7 are satisfied.*

The corollary directly follows from Proposition 3 by constructing the dependence on the shock components, z_0, z_1, z_2, as follows: First, $D(z)$ depends on z_1 only—that is, we can write $D(z) = D(z_1)$. Thus, the total amount of displacement is driven by the standard shock component. Second,

$\tilde{f}_h(z)$ depends on z_0 only—that is, we can write $\tilde{f}_h(z) = \tilde{f}_h(z_0)$. Thus, the fractions of the new Lucas tree that go to the different agents are driven by the purely transitory component of the shock. Finally, we assume that $\sum_{j\in\mathbf{J}} d_j(z_t)+\sum_{h\in\mathbf{H}} e_h(z_t) = \tau_1(z_1)+\tau_2(z_2)$, where $\tau_1(z_1)$ is a stochastic shock to endowments and dividends that is persistent and $\tau(z_2)$ is a transitory component of aggregate output.

4.2 A neoclassical growth model with heterogeneous agents

We consider a one-sector stochastic production economy with infinitely lived heterogeneous agents and we prove the existence of a recursive equilibrium with beginning-of-period cash-at-hand across agents as endogenous state.

The H infinitely lived agents have time-separable utility over consumption, supply labor inelastically, and decide each period how much to consume and how much to save in risky capital. To emphasize that there is only a single consumption good, we now denote each agent h's consumption at t by $c_{h,t}$. We allow discount factors to differ across agents and to be stochastic. Agent h's expected utility function is thus given by

$$U_h((c_{h,t})_{t=0}^{\infty}) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\prod_{k=0}^{t}\delta_h(z_k)\right)u_h\left(z_t, c_{h,t}\right)\right].$$

At each node z^t agent h has a labor endowment $l_h(z^t) = l_h(z_t)$, which he or she supplies inelastically at the market wage $w(z^t)$. There is a storage technology that uses one unit of the consumption good today to produce one unit of the capital good for the next period. We denote the investment of household h in this technology by $\alpha_h(z^t) \geqslant 0$ and the initial endowment in capital by $\alpha_h(z^{-1}) \geqslant 0$, where $\sum_{h\in\mathbf{H}} \alpha_h(z^{-1}) > 0$. At time t the household sells the capital goods accumulated from the previous period, $\alpha_h(z^{t-1})$, to the firm for a market price of $1 + r(z^t) > 0$. The price of the consumption good at each date event is normalized to one. The intertemporal budget constraint of household h at node z^t therefore reads

$$c_h(z^t) + \alpha_h(z^t) = l_h(z^t)w(z^t) + (1 + r(z^t))\alpha_h(z^{t-1}), \quad \alpha_h(z^t) \geqslant 0.$$

For simplicity we assume that there are no financial markets. As in Section 3.4 the argument can be extended to a model with financial assets and appropriate trading restrictions.

There is a single representative firm, which in each period t uses labor and capital to produce the consumption good according to a constant-returns-to-scale production function $F(z_t, K, L)$. Since the firm maximizes profits, the rate of return on capital, $1 + r(z_t)$, will always equal the marginal product of capital, $F_K(z_t, K, L)$, and the wage, $w(z^t)$, will equal the marginal product of labor, $F_L(z_t, K, L)$.

For given initial conditions, $\big(z_0, (\alpha_h(z^{-1}))_{h\in\mathbf{H}}\big)$, a *competitive equilibrium* is a collection of choices for households, $(c_h(z^t), \alpha_h(z^t))_{h\in\mathbf{H}}$, and for the representative producer, $(K(z^t), L(z^t))$, and prices,

$(r(z^t), w(z^t))$, such that households and the firm maximize and markets clear—that is to say, for all z^t

$$L(z^t) = \sum_{h=1}^{H} l_h(z^t), \quad K(z^t) = \sum_{h=1}^{H} \alpha_h(z^{t-1}). \tag{11}$$

In order to use our analysis above to show the existence of a recursive equilibrium, we need to reformulate Assumptions 1, 2, and 4. In Assumption 1 it is assumed that the agent has strictly positive endowments in the consumption good and for Assumption 4 it is crucial to assume that agents have endowments in the capital good (in every period). Instead, we want to assume that agents only have positive endowments in labor. In order to prove the existence of a recursive equilibrium and formulate a version of Assumption 4, we therefore need to redefine the endogenous state. Instead of beginning-of-period capital holdings we will take "cash-at-hand" (i.e., the sum of wages and returns to capital) across agents to be the endogenous state variable. Formally, we define the cash-at-hand of agent h at z^t to be

$$\kappa_h(z^t) = l_h(z^t)w(z^t) + (1 + r(z^t))\alpha_h(z^{t-1}).$$

This choice of state variable allows us to make natural assumptions on fundamentals. The analogue to Assumption 1 is as follows.

Assumption 8

1. *Labor endowments are bounded above and below: there are $\bar{l} > \underline{l} > 0$ such that for all $z \in \mathbf{Z}$ and all $h \in \mathbf{H}$,*

$$\bar{l} > l_h(z) > \underline{l}.$$

2. *For all $h \in \mathbf{H}$ the instantaneous discount factor is measurable in z and for any $z \in \mathbf{Z}$ it satisfies $\delta_h(z) \in (0, 1)$.*

3. *The Bernoulli functions, $u_h : \mathbf{Z} \times \mathbb{R}_{++} \to \mathbb{R}$, $h \in \mathbf{H}$, are measurable in z and strictly increasing, strictly concave, and continuously differentiable in c. For each $z \in \mathbf{Z}$ they satisfy a strong Inada condition: Along any sequence $c^n \to 0$, $\sup_{z \in \mathbf{Z}} u_h(z, c^n) \to -\infty$. Moreover, utility is bounded above—that is, there exists a $\bar{u}$ such that for all $h \in \mathbf{H}$, $u_h(z, c) \leqslant \bar{u}$ for all $z \in \mathbf{Z}$, $c \in \mathbb{R}_{++}$.*

To simplify notation we define $u'_h(z, c) = \frac{\partial u_h(z,c)}{\partial c}$. Instead of the rather abstract Assumption 2 we now have the following.

Assumption 9

1. *The production function, $F(z, K, L)$ is measurable in z and continuously differentiable in (K, L).*

2. *For each $z \in \mathbf{Z}$, $F(z, .)$ is concave and increasing in (K, L) and it exhibits constant returns to scale.*

3. *For each* $z \in \mathbf{Z}$ *and for each* $L > 0$ *we have* $\lim_{K \to 0} F_K(z, K, L) = +\infty$ *and* $F(z, 0, L) = 0$*; for each* $K > 0$ *we have* $\lim_{L \to 0} F_L(z, K, L) = +\infty$ *and* $F(z, K, 0) = 0$.

4. *There is some* $\bar{K} < +\infty$ *such that* $F(z, K, \sum_{h \in \mathbf{H}} l_h(z)) < K$ *for all* $K > \bar{K}$*, and all* $z \in \mathbf{Z}$.

Assumption 9 readily implies that in equilibrium aggregate production is always bounded above by some $\overline{\kappa}$ if the initial aggregate cash-at-hand is below $\overline{\kappa}$. In applications researchers often assume Cobb–Douglas production with a multiplicative TFP shock. This is consistent with our assumptions as long as depreciation is positive at all shocks. Since aggregate labor used in production is a function of the shock alone, we can write the production function and its derivatives as

$$f(z, K) = F(z, K, \sum_{h=1}^{H} l_h(z)), \;\; f_K(z, K) = F_K(z, K, \sum_{h=1}^{H} l_h(z)), \;\; f_L(z, K) = F_L(z, K, \sum_{h=1}^{H} l_h(z)).$$

Since we assume that agents have no endowments in the capital good, we need to make an additional assumption to ensure that in any equilibrium aggregate capital is always bounded away from zero. We make the following assumption.

ASSUMPTION 10 *There is a* $\underline{K} > 0$ *and an* $\epsilon > 0$ *such that for each agent* h *and all* $K \geqslant \underline{K}$,

$$\inf_{z \in \mathbf{Z}} \quad \left[-u_h'\left(z, l_h(z) f_L(z, K) + \tfrac{K}{H} f_K(z, K) - \underline{K}/H\right) + \right.$$
$$\left. \mathbb{E}_z\left[\delta_h(z') f_K(z', \underline{K}) u_h'\left(z', f_K(z', \underline{K})\underline{K}/H + l_h(z') f_L(z', \underline{K})\right)\right]\right] > \epsilon.$$

Although this appears to be a complicated joint assumption on utility and production, it can be verified as holding in standard settings. This assumption guarantees that aggregate capital will always be above $\underline{K}$. Together with the assumption of strictly positive labor endowments this assumption also implies a lower bound on each individual's cash-at-hand, which we denote by $\underline{\kappa} > 0$.

We define the endogenous part of the state space to be

$$\mathbf{K} = \{\kappa \in \mathbb{R}_+^H : \sum_{h \in \mathbf{H}} \kappa_h \leqslant 2\bar{\kappa}, \text{ and } \kappa_{hl} \geqslant \frac{1}{2}\underline{\kappa} \text{ for all } h \in \mathbf{H}\} \tag{12}$$

and assume, as above, that the shock space can be decomposed into three complete, separable metric spaces, $\mathbf{Z} = \mathbf{Z}_0 \times \mathbf{Z}_1 \times \mathbf{Z}_2$, with Borel σ-algebra $\mathcal{Z} = \mathcal{Z}_0 \otimes \mathcal{Z}_1 \otimes \mathcal{Z}_2$. For each $i = 0, 1, 2$ there is a measure μ_{z_i} on $\mathbf{Z}_i$ and there are conditional densities $r_{z_0}(z_0'|z, z_1')$, $r_{z_1}(z_1'|z)$, and $r_{z_2}(z_2'|z, z_0', z_1')$ such that for any $B \in \mathcal{Z}$ we have

$$\mathbb{P}(B|z) = \int_{\mathbf{Z}_1} \int_{\mathbf{Z}_0} \int_{\mathbf{Z}_2} \mathbb{1}_B(z') r_{z_2}(z_2'|z, z_0', z_1') r_{z_0}(z_0'|z, z_1') r_{z_1}(z_1'|z) d\mu_{z_2}(z_2') d\mu_{z_0}(z_0') d\mu_{z_1}(z_1').$$

To ensure continuity of the state transition in Assumption 3.1 we assume that the shock z_0 is purely transitory, has a continuous density, and only affects agents' endowments in labor. Moreover, given z_1 and z_2, there is a C^1-diffeomorphism from $\mathbf{Z}_0$ to a subset of possible labor endowments. More precisely, we retain Assumptions 4.1 and 4.2 above and replace Assumption 4.3 and Assumption 5 by the following assumption.

Assumption 11

1. *For each agent h, and all $(z_1, z_2) \in Z_1 \times Z_2$, $(l_h(., z_1, z_2))_{h \in \mathbf{H}}$ is a C^1-diffeomorphism from $\mathbf{Z}_0$ to a bounded subset of $\mathbb{R}_+^H$ with a non-empty interior. All other fundamentals are independent of z_0.*

2. *Conditionally on next period's z_1', the shock z_2' is independent of both z_0' and the current shock z. Conditionally on z_1', the measure $\mu_{z_2}(.|z_1')$ is absolutely continuous with respect to some atomless probability measure on $\mathbf{Z}_2$ so that we can write the density as $r_{z_2}(z_2'|z, z_0', z_1') = r_{z_2}(z_2'|z_1')$. Moreover, $f(z, K)$ does not depend on z_2 and, for each agent h, $l_h(z)$ does not depend on z_2.*

We thus assume that $\delta_h(z_t)$ and $u_h(z_t, .)$ can possibly depend on z_2. Moreover, the probabilities over future realizations of z_1 can clearly depend on z_2. In this case, z_2 can be interpreted as a "news shock".

As above, we can now take the state-space to consist of shocks 1 and 2 as well as the endogenous state. That is to say, we take

$$\mathbf{S} = \mathbf{Z}_1 \times \mathbf{Z}_2 \times \mathbf{K}$$

with Borel σ-algebra $\mathcal{S}$. We have the following theorem.

Proposition 4 *Under Assumptions 8, 9, 10, 4.1, 4.2, and 11 there exists a recursive equilibrium.*

The proof of this proposition is along the lines of the proofs of Proposition 2 and Theorem 1. However, since we define the endogenous state differently, some key parts are different. A complete proof can be found in the appendix.

As for the case of the Lucas tree model, it is useful to give one concrete specification of shocks that satisfies our assumptions.

Corollary 2 *Suppose each agent's labor endowments can be written as the sum of an i.i.d. component that has a continuous density over a compact subset of $\mathbb{R}_{++}$, and of a component that depends on some shock z_1 that follows a Markov process. Also suppose that production functions and utility functions depend on this shock z_1. If there exists a shock z_2 that is independent of the past realization of (z_1, z_2) but might depend on the current z_1, and if this shock does not affect any fundamentals except possibly discount factors, utility, and transition probabilities then the Assumptions 4.1, 4.2, and 11 are satisfied.*

5 Conclusion

We prove the existence of recursive equilibria in general stochastic production economies with infinitely lived agents and incomplete markets. In order to do so, we have to make some nonstandard assumptions on the stochastic process of economic fundamentals.

Most importantly, we need to assume that there are atomless shocks to fundamentals. In contrast, in many applications exogenous shocks follow a Markov chain with finite support. However, such a discrete-shock process is often just an approximation to a true data-generating process with atomless innovations (e.g., following Tauchen and Hussey (1991)). In this case, one should be more concerned with the existence of an epsilon-equilibrium of the discrete-shock model and its relation to an exact equilibrium for the continuous-shock model. This question can only be posed if the existence of an exact recursive equilibrium can be guaranteed.

In addition, we need to guarantee, by Assumption 4, that agents' current choices lead to a non-degenerate distribution over the endogenous state next period. This is in contrast to many standard models in which current choices pin down next period's endogenous state deterministically. In stochastic games, however, it is well known that a so-called deterministic transition creates problems for the existence of Markov equilibria (see, e.g., Levy (2013)).

Moreover, Levy and McLennan (2015) provide an example of a stochastic game that illustrates that continuity assumptions along the line of our Assumption 4 are not sufficient to guarantee the existence of a Markov equilibrium and that a version of Assumption 5 is needed as well. Our stylized example of nonexistence in Section 2.3 violates Assumptions 4 and 5 above.[10] To see that Assumption 5 does not suffice to ensure existence, note that a special case of the assumption is to assume that one component of the shock does not affect fundamentals (a sunspot) and is i.i.d. with atomless distribution. Equilibrium prices may then depend on the realization of this shock, but irrespective of what one assumes about the distribution of prices, the same argument as in Section 2.3 implies that there can never be an equilibrium with positive storage. Zero storage, however, entails that—independently of the realization of the sunspot—the price in shock 3 is uniquely determined by the shock in the previous period, implying the nonexistence of a recursive equilibrium. In contrast to Assumption 5, one can easily verify that Assumption 4 restores existence in the example.[11] As long as there is a non-atomic shock to the endowments of the capital good a recursive equilibrium exists in our example. If one considers a sequence of economies along which the variance of the shock converges to zero one can obtain the existence of a competitive equilibrium in the limit, but this equilibrium is not recursive.

While we have to make some strong assumptions, our paper provides the only result in the literature that ensures the existence of a recursive equilibrium in any variation of the model. Therefore, even if the assumptions do not hold for a specific tractable formulation used in an application, it is useful to understand under which additional assumptions existence can be obtained. It is the subject of further research to examine whether the general existence of a recursive equilibrium can

[10] It also violates Assumption 1, yet it is easy to see that Assumption 1 alone cannot restore existence in the example; the specific endowments and preferences were simply chosen to make the examples as simple as possible.

[11] Unfortunately, there are many other perturbations to fundamentals that also restore existence. As Citanna and Siconolfi (2008) point out (for OLG economies, but the argument also applies to examples of nonexistence in economies with infinitely lived agents), all examples for which the economy decomposes into several two-period economies suffer from the shortcoming that they are not robust—perturbations in endowments restore the existence of a recursive equilibrium.

be established without some version of Assumptions 4 and 5.

Appendix: Proofs

Proof of Proposition 1.

Note that if the conditions in the lemma are satisfied then there exist $((\bar{x}_{h,t}, \bar{\alpha}_{h,t})_{h\in\mathbf{H}}, \bar{p}_t)_{t=0}^{\infty}$ such that markets clear, budget equations hold, and there exist multipliers $\nu_h(z^t)$ and $\xi_h(z^t)$ such that the following first order conditions hold for each agent $h \in \mathbf{H}$ and all z^t.

$$\begin{aligned} D_x u_h(z_t, \bar{x}_h(z^t)) - \nu_h(z^t)\bar{p}(z^t) + \xi_h(z^t) &= 0 && (13)\\ \bar{x}_h(z^t) \perp \xi_h(z^t) &\geqslant 0 && (14)\\ \bar{\alpha}_h(z^t) \perp \left(-\nu_h(z^t)\bar{p}(z^t)A_h^0(z_t) + \delta\mathbb{E}_{z^t}\left[\nu_h(z^{t+1})\bar{p}(z^{t+1})A_h^1(z_{t+1})\right]\right) &\geqslant 0. && (15)\end{aligned}$$

It suffices to show that these conditions are sufficient for $(\bar{x}_{h,t}, \bar{\alpha}_{h,t})$ to be a solution to the agents' infinite horizon problem. Following Duffie et al. (1994), assume that for any agent h, given prices, a budget feasible policy $(\bar{x}_{h,t}, \bar{\alpha}_{h,t})$ satisfies (13)–(15). Suppose there is another budget feasible policy $(x_{h,t}, \alpha_{h,t})$. Since the value of consumption in 0 only differs by the value of production plans, concavity of $u_h(z, .)$ together with the gradient inequality implies that

$$\begin{aligned} u_h(z_0, \bar{x}_h(z^0)) &\geqslant u_h(z_0, x_h(z^0)) + D_x u_h(z_u, \bar{x}_h(z^0))(\bar{x}_h(z^0) - x_h(z^0)) && (16)\\ &\geqslant u_h(z_0, x_h(z^0)) + \nu_h(z^0)\bar{p}(z^0)(\bar{x}_h(z^0) - x_h(z^0))\\ &= u_h(z_0, x_h(z^0)) + \nu_h(z^0)\bar{p}(z^0)A_h^0(z_0)(\alpha_h(z^0) - \bar{\alpha}_h(z^0)).\end{aligned}$$

We show by induction, that for any T it holds that

$$\begin{aligned} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\delta^t u_h(z_t, \bar{x}_h(z^t))\right] \geqslant\; & \mathbb{E}_0\left[\textstyle\sum_{t=0}^{T}\delta^t u_h(z_t, x_h(z^t))\right] + \mathbb{E}_0\left[\textstyle\sum_{t=T+1}^{\infty}\delta^t u_h(z_t, \bar{x}_h(z^t))\right] + && (17)\\ & \delta^T\mathbb{E}_0\left[\nu_h(z^T)p(z^T)A_h^0(z_T)(\alpha_h(z^T) - \bar{\alpha}_h(z^T))\right].\end{aligned}$$

By (16), this inequality holds for $T = 0$. To obtain the induction step when $\bar{\alpha}_j > 0$, we use the first order conditions to substitute $\delta\mathbb{E}_{z^{t-1}}\left[\nu_h(z^t)\bar{p}(z^t)a_{hj}^1(z_t)(\alpha_{hj}(z^{t-1}) - \bar{\alpha}_{hj}(z^{t-1}))\right]$ for

$$\nu_h(z^{t-1})\bar{p}(z^{t-1})a_{hj}^0(z_{t-1})(\alpha_{hj}(z^{t-1}) - \bar{\alpha}_{hj}(z^{t-1})),$$

and then apply the budget constraint and the law of iterated expectations. When $\bar{\alpha}_j = 0$, it is clear that $\alpha_j \geqslant \bar{\alpha}_j$ and since

$$\delta\mathbb{E}_{z^{t-1}}\left[\nu_h(z^t)\bar{p}(z^t)a_{hj}^1(z_t)\right] \geqslant \nu^h(z^{t-1})\bar{p}(z^{t-1})a_{hj}^0(z_{t-1}),$$

the induction step follows.

The second term on the right hand side of (17) will converge to zero as $T \to \infty$ since u_h is bounded above by Assumption 1 and below for the following reason: $M_{h1}(s) = \frac{\partial u_h(z,x_h)}{\partial x_1}$ is bounded above, thus by the strong Inada condition x_1 is bounded below by some $\underline{x}_1$, and therefore utility is bounded below by $u_h(\underline{x}_1, 0, \ldots, 0)$. The third term will converge to zero because the M-functions are assumed to be bounded above and production is bounded by Assumption 2. □

Proof of Lemma 1.

The proof is analogous to the proof of Lemma 1 in Duggan (2012). □

Proof of Lemma 2.

For any $\bar{M} \in \mathbf{M}^b$ and $s = (z, \kappa) \in \mathbf{S}$, we define the following compact sets:

$$\widetilde{\mathbf{Y}}(s) = \{y \in \mathbf{Y}(z) : y + \sum_{h \in \mathbf{H}} (e_h(z) + \kappa_h) \geqslant 0\},$$

$$\widetilde{\mathbf{C}}(s) = \{x_h \in \mathbf{C} : \frac{1}{2}x_h - \sum_{h \in \mathbf{H}} (e_h(z) + \kappa_h) \in \mathbf{Y}(z), x_{h1} \geqslant \underline{c}_{\bar{M}}\},$$

$$\mathbf{A} = \{\alpha_h \in \mathbb{R}^J_+ : f_{hl}(z') + \sum_{j \in \mathbf{J}} a^1_{hlj}(z')\alpha_{hj} \leqslant 2\bar{\kappa} \text{ for all } h \in \mathbf{H}, l \in \mathbf{L}, z' \in \mathbf{Z}\}.$$

The lower bound on consumption in good 1, $\underline{c}_{\bar{M}}$, in the definition of $\widetilde{\mathbf{C}}(s)$ will generally be different from $\underline{c}$ as defined in Section 3.1, but for any given $\bar{M}$ its existence is guaranteed by Assumption 1. To ensure compactness of $\mathbf{A}$ it is without loss of generality to assume that for each agent h and each storage technology j there is a commodity l and a shock z' such that $a^1_{h,l,j}(z') > 0$. If this is not the case, it is always optimal for all h to set $\alpha_{h,j} = 0$.

For a given $\eta > 0$ we define the truncated price set $\Delta^{L-1}_\eta = \{p \in \mathbb{R}^L_+ : \sum_{l=1}^L p_l = 1, p_1 \geqslant \eta\}$ and for each agent $h = 1, \ldots, H$ the choice correspondence

$$\Phi^h_\eta : \Delta^{L-1}_\eta \times \mathbf{\Xi} \rightrightarrows \widetilde{\mathbf{C}}(s) \times \mathbf{A}.$$

by

$$\begin{aligned} \Phi^h_\eta(p, \alpha^*) = \arg \max_{x_h \in \tilde{\mathbf{C}}(s), \alpha_h \in \mathbf{A}} & E^{\bar{M}}_h(s, x_h, \alpha_h, \alpha^*) \text{ s.t.} \\ & -p \cdot (x_h - e_h(z) - \kappa_h + A^0_h(z)\alpha_h) \geqslant 0 \end{aligned}$$

By a standard argument, the correspondence Φ is convex-valued, non-empty valued, and upper-hemicontinuous. Define the producer's best response $\Phi^{H+1}_\eta : \Delta^{L-1}_\eta \rightrightarrows \tilde{\mathbf{Y}}(s)$ by

$$\Phi^{H+1}_\eta(p) = \arg \max_{y \in \tilde{\mathbf{Y}}(s)} p \cdot y$$

and define a price player's best response,

$$\Phi^0_\eta : (\tilde{\mathbf{C}}(s) \times \mathbf{A})^H \times \tilde{\mathbf{Y}}(s) \rightrightarrows \Delta^{L-1}_\eta$$

by

$$\Phi_{\eta}^{0}((x_h,\alpha_h)_{h\in\mathbf{H}},y) = \arg\max_{p\in\Delta_{\eta}^{L-1}} p\cdot\left(\sum_{h\in\mathbf{H}}(x_h - e_h(z) - \kappa_h + A_h^0(z)\alpha_h) - y\right).$$

It is easy to see that this correspondence is also upper-hemicontinuous, non-empty, and convex valued. Finally, define

$$\Phi^{H+2} : \mathbf{A}^H \rightrightarrows \mathbf{\Xi}$$

by

$$\Phi^{H+2}(\alpha) = \arg\min_{\alpha^*\in\mathbf{\Xi}} \|\alpha - \alpha^*\|_2.$$

By Kakutani's fixed-point theorem, the correspondence $\left(\bigtimes_{h=0}^{H+1}\Phi_{\eta}^{h}\right)\times\Phi^{H+2}$ has a fixed point, which we denote by $(\bar{x},\bar{\alpha},\bar{y},\bar{\alpha}^*,\bar{p})$.

Since by budget feasibility we must have

$$\bar{p}\cdot\left(\sum_{h\in\mathbf{H}}(\bar{x}_h - e_h(z) - \kappa_h + A_h^0(z)\bar{\alpha}_h) - \bar{y}\right) \leqslant 0,$$

optimality of the price player implies that for sufficiently small $\eta > 0$ the upper bound imposed by requiring $x\in\tilde{\mathbf{C}}(s)$ and the upper bound on production will both never bind. Consumption solves the agent's problem for all $x\in\mathbf{C}$ and production maximizes profits among all $y\in\mathbf{Y}(z)$. In addition, Assumption 2 implies that the upper bound on each α_h cannot be binding and that in fact $\bar{\alpha}=\bar{\alpha}^*$.

Finally, there must be some $\epsilon > 0$ such that for all $\eta < \epsilon$ the fixed point must satisfy that $\bar{p}_1 \geqslant \epsilon$. This is true because all commodities must either be consumed, used as an input for intra-period production, or stored. If $\bar{p}_1 < \epsilon$ there must be some other commodity $l \neq 1$ with $\frac{\bar{p}_l}{\bar{p}_1} > \frac{1-\epsilon}{(L-1)\epsilon}$. But for sufficiently small $\epsilon > 0$ the (relative) price of this commodity is so high that it is not consumed—because marginal utility of good 1 is bounded away from zero in $\tilde{\mathbf{C}}(s)$ and marginal utility of commodity l is finite (as utility is assumed to be continuously differentiable on $\mathbf{C}=\mathbf{R}_{++}\times\mathbf{R}_{+}^{L-1}$). Furthermore, good l can neither be used for (constant-returns-to-scale) intratemporal production, nor for (linear) storage—the agent who stores it could eventually increase his utility by selling a small fraction of this commodity and increasing his consumption of commodity 1. Therefore there is some $\epsilon > 0$ such that for $\eta < \epsilon$ the price player chooses a price with $\bar{p}_1 \geqslant \epsilon$ and a standard argument gives that

$$\sum_{h\in\mathbf{H}}(\bar{x}_h - e_h(z) - \kappa_h + A_h^0(z)\bar{\alpha}_h) = \bar{y}.$$

This proves the lemma. □

Proof of Lemma 3.

For given $s\in\mathbf{S}$ the set of allocations $x\in\mathbb{R}_{+}^{HL}$, $\alpha\in\mathbf{\Xi}$, $y\in\mathbf{Y}(z)$, and prices $p\in\Delta^{L-1}$ satisfying (6), (7), and (8) can be described as solutions to a system of equations and inequalities, compare the proof of Proposition 1. Moreover, the correspondence $s\rightrightarrows\mathbf{N}_{\bar{M}}(s)$ is (non-empty) compact-valued.

Applying results from Chapter 18 in Aliprantis and Border (2006) and Himmelberg (1975), it is easy to show that the correspondence $s \rightrightarrows \mathbf{P}_{\bar{M}}(s)$ is weakly measurable.

By the selection theorem of Kuratowski and Ryll-Nardzewski, $\mathbf{P}_{\bar{M}}$ has a measurable selector (see Theorem 18.13 in Aliprantis and Border(2006)). Consequently, the map $M \rightrightarrows \mathbf{R}^{co}(M)$ is non-empty valued, and obviously it is also convex-valued. Take $M^n \to M$ as $n \to \infty$, $M^n, M \in \mathbf{M}$ and $v^n \to v$ such that $v^n \in \mathbf{R}^{co}(M^n)$ for each n. We assume that both sequences converge in the weak* topology $\sigma(L^m_\infty, L^m_1)$. We need to show $v \in \mathbf{R}(M)$. Since for a given s each $E^M_h(s,.)$ is jointly continuous in (x, α, α^*) and M and since the equilibrium conditions can be expressed as weak inequalities the correspondence $M \rightrightarrows \mathbf{P}_M(s)$ has a closed graph. Theorem 17.35 (2) in Aliprantis and Border (2006) implies that the correspondence $M \rightrightarrows \mathbf{P}^{co}_M(s)$ has a closed graph as well. Moreover, since $\mathbf{S}$ is a finite measure space Mazur's lemma implies that there exists a sequence $\hat{v}^n$ of finite convex combinations of $\{v^p : p = 1, 2, ...\}$ such that some subsequence $\hat{v}^n$ converges to v almost surely, i.e., $\hat{v}^n(s) \to v(s)$ for every $s \in \mathbf{S} \backslash S_1$ where the set S_1 is of measure zero. Given the closed graph of $M \rightrightarrows \mathbf{P}^{co}_M(s)$ we must have that for any $\epsilon > 0$, for sufficiently large n the set $\mathbf{P}^{co}_{M^n}(s)$ is a subset of an ϵ-neighborhood of $\mathbf{P}^{co}_M(s)$. Therefore for any $\epsilon > 0$ there is a k_0 such that for all $k > k_0$, $\hat{v}_k(s)$ is within an ϵ-neighborhood of $\mathbf{P}^{co}_M(s)$. Therefore $v(s)$ is in an ϵ-neighborhood of $\mathbf{P}^{co}_M(s)$ but since ϵ is arbitrary we must have $v(s) \in \mathbf{P}^{co}_M(s)$ for any $s \in \mathbf{S} \backslash S_1$. This proves that $\mathbf{R}^{co} : M \rightrightarrows L^+_\infty$ has a closed graph. Similarly it can be shown that $M \rightrightarrows \mathbf{R}^{co}(M)$ is weak* closed-valued. □

Proof of Lemma 4.

Assumption 3 states that there is a sub-σ-algebra $\mathcal{G}$ of $\mathcal{S}$ such that $\mathcal{S}$ has no $\mathcal{G}$-atom. Defining

$$\mathcal{I}^{\mathcal{S},\mathcal{G}}_F = \{\mathbb{E}[f|\mathcal{G}] : f \text{ is an } \mathcal{S}\text{-measurable selection of } F\},$$

it follows from Theorem 5 in Dynkin and Evstigenev (1976) that since $\mathcal{S}$ has no $\mathcal{G}$-atom, it must hold that $\mathcal{I}^{\mathcal{S},\mathcal{G}}_F = \mathcal{I}^{\mathcal{S},\mathcal{G}}_{co(F)}$. Therefore, for each measurable selection of $\mathbf{F}^{co}$, M, there is a measurable selection, $\hat{M}$, of $\mathbf{F}$ such that $\mathbb{E}[M|\mathcal{G}] = \mathbb{E}\left[\hat{M}|\mathcal{G}\right]$. Therefore we must have for each h and all (s, α) that

$$\begin{aligned}
\int_{\mathbf{S}} M_h(s') A^1_h(z') d\mathbb{Q}(s'|s,\alpha) &= \int_{\mathbf{S}} M_h(s') A^1_h(z') q(s'|s,\alpha) d\lambda(s') = \int_{\mathbf{S}} \mathbb{E}\left[M_h A^1_h q_{s,\alpha}|\mathcal{G}\right] d\lambda(s') \\
&= \int_{\mathbf{S}} \mathbb{E}\left[M_h|\mathcal{G}\right] A^1_h(z') q(s'|s,\alpha) d\lambda(s') = \int_{\mathbf{S}} \mathbb{E}\left[\hat{M}_h|\mathcal{G}\right] A^1_h(z') q(s'|s,\alpha) d\lambda(s') \\
&= \int_{\mathbf{S}} \hat{M}_h(s') A^1_h(z') d\mathbb{Q}(s'|s,\alpha).
\end{aligned}$$

This proves the result. □

Proof of Lemma 5.

The proof of Lemma 2 implies that there exists some $\bar{c}$ such that whenever $((x_h)_{h \in \mathbf{H}}, p) \in \mathbf{N}_{\bar{M}}(s)$ for some $\bar{M} \in L^+_\infty$ and some $s \in \mathbf{S}$ then $x_{hl} \leqslant \bar{c}$ for all h, l. This, together with Assumption 1 implies that there is an $\epsilon > 0$ such that if $(x(z^t))$ is the consumption choice of some agent in any competitive equilibrium then

$$u_h(z_t, \tilde{x}) + \mathbb{E}_{z^t}\left[\sum_{i=1}^{\infty} \delta^i u_h(z_{t+i}, (1-\epsilon)x(z^{t+i}))\right] > \mathbb{E}_{z^t}\left[\sum_{i=0}^{\infty} \delta^i u_h(z_{t+i}, x(z^{t+i}))\right],$$

where $\tilde{x} = (x_1(z^t) + 1, (1-\epsilon)x_2(z^t), \ldots, (1-\epsilon)x_L(z^t))$. An upper bound on relative equilibrium prices of goods that are not used as inputs to intra-temporal production is then given by $\frac{2}{\underline{\omega}\epsilon}$; if the price of any commodity relative to good 1 is above this threshold, then some agent can sell a fraction ϵ of this commodity, consume one unit more of commodity 1 and increase his lifetime utility. Assumption 2 implies that the relative price of any input to production is bounded above by the zero profit condition. Taken together this implies that there is some p^u such that in any competitive equilibrium it must be that $\frac{p_l}{p_1} < \frac{1}{2}p^u$.

Define a weak* compact and convex set

$$\mathbf{M}^0 = \{M \in L_\infty^+ : M_{h1} \leqslant \bar{m}, M_{hl} \leqslant p^u \bar{m},\ l = 2, \ldots, L \ a.s. \text{ for all } h \in \mathbf{H}\},$$

where $\bar{m}$ is defined in (1). For any set $\mathbf{M} \subset L_\infty^+$ define a correspondence $s \rightrightarrows \mathbf{P}_{\mathbf{M}}^{co}(s)$ by

$$\mathbf{P}_{\mathbf{M}}^{co}(s) = \text{conv}\left(\bigcup_{M \in \mathbf{M}} \mathbf{P}_M(s)\right) \text{ for all } s \in \mathbf{S},$$

and denote by $\mathbf{R}^{co}(\mathbf{M})$ the set of measurable selections of $\mathbf{P}_{\mathbf{M}}^{co}$. We construct the set $\mathbf{M}^*$ inductively by defining, for each $i = 0, 1, 2 \ldots$, $\mathbf{M}^{i+1} = \mathbf{R}^{co}(\mathbf{M}^i) \cap \mathbf{M}^0$.

Note that by the construction of $\mathbf{M}^i$ and by Lemma 4 any element of $\mathbf{M}^i$, if it exists, consists of equilibrium marginal utilities for an artificial $(i+1)$-period economy where in the last period agents have some continuation utility $M^0 \in \mathbf{M}^0$.

Defining $M^0 = 0$, it is clear from the above construction of $\mathbf{M}^0$ that whenever $M^1 \in \mathbf{R}^{co}(M^0)$ we must have $M^1 \in \mathbf{M}^0$. From the above argument it follows that whenever $M^2 \in \mathbf{R}^{co}(M^1)$ we must have $M^2 \in \mathbf{M}^0$, and in fact that each M^i defined recursively in this manner will lie in $\mathbf{M}^0$. Therefore each $\mathbf{M}^i$ is non-empty. Obviously each $\mathbf{M}^i$ is also convex. By the same argument as in the proof of Lemma 3 each $\mathbf{M}^i$ is weak* closed and hence as a subset of $\mathbf{M}^0$ it is weak* compact. Obviously we have $\mathbf{M}^1 \subset \mathbf{M}^0$ and if $\mathbf{M}^i \subset \mathbf{M}^{i-1}$ it must follow that $\mathbf{M}^{i+1} \subset \mathbf{M}^i$. Therefore we must have for any $i = 0, 1, \ldots$ that for all $M \in \mathbf{M}^i$

$$\mathbf{R}^{co}(M) \cap \mathbf{M}^0 \subset \mathbf{M}^i.$$

Assumption 1, together with discounting and the construction of the upper bound p^u guarantee that for T sufficiently large, if $\bar{M} \in \mathbf{M}^T$ there can be no solution to (6), (7), and (8) with $x_1 \leqslant \frac{1}{2}\underline{c}$, or with $\frac{p_l}{p_1} > p^u$ for some $l = 2, \ldots, L$. Therefore we have $\mathbf{R}^{co}(\bar{M}) \subset \mathbf{M}^0$ and we can take $\mathbf{M}^* = \mathbf{M}^T$.
□

Proof of Lemma 6.

We first show that under Assumptions 4 and 5 , $\mathbb{Q}(.|s,\alpha)$ satisfies Assumption 3.1. By Assumption 5, it suffices to show norm-continuity for the marginal transition function on $\underline{\mathcal{S}}$, i.e., that for any sequence $\alpha^n \in \mathbf{\Xi}$ with $\alpha^n \to \alpha^0 \in \mathbf{\Xi}$

$$\sup_{B\in\underline{\mathcal{S}}} |\mathbb{Q}_{\underline{s}}(B|s,\alpha^n) - \mathbb{Q}_{\underline{s}}(B|s,\alpha^0)| \to 0$$

for all $s \in \mathbf{S}$. To show this, we first define for given $(z, z_1') \in \mathbf{Z}\times\mathbf{Z}_1$ and $\alpha \in \mathbf{\Xi}$ a C^1-diffeomorphism $g_{(z,z_1',\alpha)}$ that maps $\mathbf{Z}_0$ into its range $\bar{\mathbf{K}}_{(z,z_1',\alpha)} = g_{(z,z_1',\alpha)}(\mathbf{Z}_0) \subseteq \mathbf{K}$ with $g_{(z,z_1',\alpha)}(z_0') = (f_h(z_0', z_1') + A_h^1(z_1')\alpha_h)_{h\in\mathbf{H}}$ and a density

$$r_\kappa(\kappa'|z,z_1',\alpha) := \begin{cases} r_{z^0}(g^{-1}_{(z,z_1',\alpha)}(\kappa')|z,z_1') \cdot \left|J(g^{-1}_{(z,z_1',\alpha)}(\kappa'))\right| & \text{if } \exists\ z_0 : g_{(z,z_1',\alpha)}(z_0) = \kappa' \\ 0 \quad \text{otherwise,} \end{cases}$$

where $|J(\cdot)|$ denotes the determinant of the Jacobian.

Denoting by μ_κ the Lebesgue measure defined on the σ-algebra of Borel subsets of $\mathbf{K}$, for $B \in \underline{\mathcal{S}}$ and $\alpha^n \in \mathbf{\Xi}$ with $\alpha^n \to \alpha^0 \in \mathbf{\Xi}$, we have

$$\begin{aligned}
\mathbb{Q}_{\underline{s}}(B|s,\alpha^n) &= \int_{\mathbf{Z}_1}\int_{\mathbf{Z}_0} \mathbb{1}_B\left[z_1', (f_h(z') + A_h^1(z')\alpha_h^n)_{h\in\mathbf{H}}\right] r_{z_0}(z_0'|z,z_1') r_{z_1}(z_1'|z) d\mu_{z_0}(z_0') d\mu_{z_1}(z_1') \\
&= \int_{\mathbf{Z}_1}\int_{\mathbf{Z}_0} \mathbb{1}_B\left[z_1', g_{(z,z_1',\alpha^n)}(z_0')\right] r_{z_0}(z_0'|z,z_1') r_{z_1}(z_1'|z) d\mu_{z_0}(z_0') d\mu_{z_1}(z_1') \\
&= \int_{\mathbf{Z}_1}\int_{\bar{\mathbf{K}}_{(z,z_1',\alpha^n)}} \mathbb{1}_B\left[z_1', \kappa'\right] r_\kappa(\kappa'|z,z_1',\alpha^n) r_{z_1}(z_1'|z) d\mu_\kappa(\kappa') d\mu_{z_1}(z_1') \\
&= \int_{\mathbf{Z}_1}\int_{\mathbf{K}} \mathbb{1}_B\left[z_1', \kappa',\right] r_\kappa(\kappa'|z,z_1',\alpha^n) r_{z_1}(z_1'|z) d\mu_\kappa(\kappa') d\mu_{z_1}(z_1') \\
&= \int_B r_\kappa(\kappa'|z,z_1',\alpha^n) r_{z_1}(z_1'|z) d\mu_\kappa(\kappa') d\mu_{z_1}(z_1'),
\end{aligned}$$

where we used Fubini's theorem for the first equality and the change of variables theorem for the third equality. Since $(f_h(.,z_1')))_{h\in\mathbf{H}}$ is a C^1-diffeomorphism, the set $(f_h(\partial\mathbf{Z}_0, z_1'))_{h\in\mathbf{H}}$ is of measure zero and it follows that $r_\kappa(\kappa'|z,z_1',\alpha^n) \to r_\kappa(\kappa'|z,z_1',\alpha^0)$ for almost all κ'. By Scheffe's lemma we then obtain norm-continuity.

For all (s,α) the marginal distribution of $\mathbb{Q}(.|s,\alpha)$ on $\underline{\mathbf{S}}$ is absolutely continuous with respect to the product measure $\eta = \mu_\kappa \times \mu_{z_1}$ and has a Radon–Nikodym derivative $q_{\underline{s}}(z_1',\kappa'|s,\alpha) = r_\kappa(\kappa'|z,z_1',\alpha) r_{z_1}(z_1'|z)$. Defining the measure $\lambda(.)$ by

$$\lambda(B) = \int_{\underline{\mathbf{S}}}\int_{\mathbf{Z}_2} \mathbb{1}_B[\underline{s}, z_2] r_{z_2}(z_2|z_1) d\mu_{z_2}(z_2) d\eta(\underline{s})$$

and taking $\mathcal{G} = \underline{\mathcal{S}} \otimes \{\varnothing, \mathbf{Z}_2\}$, Assumption 5 together with Proposition 2 in He and Sun (2017) then implies that $\mathbb{Q}(.|s,\alpha)$ satisfies Assumptions 3.2 and 3.3. $\square$

Proof of Proposition 4.

Define

$$\mathbf{\Xi} = \{\alpha \in \mathbb{R}^H_+ : \left(\alpha_h f_K(z', \sum_i \alpha_i) + l_h(z') f_L(z', \sum_i \alpha_i)\right)_{h \in \mathbf{H}} \in \mathbf{K} \text{ for all } z' \in \mathbf{Z}\}.$$

Our definition of $\mathbf{K}$ implies that for any sequence of $\kappa(z^t) \in \mathbf{K}$ each individual's budget-feasible consumption must be bounded above and hence there is a lower bound $\underline{m} > 0$ on each individual's marginal utility. Moreover, for any such a sequence an individual will have income bounded away from zero and by the same argument as in Section 3 there is an upper bound on marginal utility, which we denote by $\bar{m}$. As in Proposition 1 we can characterize competitive equilibrium by marginal utilities.[12] For this, let $\widetilde{\mathbf{M}}$ be the set of functions in L^m_∞ that are essentially bounded below by $\underline{m}$. Given any $\bar{M} = (\bar{M}^1, \ldots, \bar{M}^H) \in \widetilde{\mathbf{M}}$ we define

$$E^{\bar{M}}_h(s, c_h, \alpha_h, \alpha^*) = u_h(z, c_h) + \delta \mathbb{E}_s \left[\bar{M}^h(s') f_K(z', \sum_{i \in \mathbf{H}} \alpha^*_i) \alpha_h \right] \tag{18}$$

with

$$s' = \left(z', \left(\alpha^*_h f_K(z', \sum_i \alpha^*_i) + l_h(z') f_L(z', \sum_i \alpha^*_i) \right)_{h \in \mathbf{H}} \right).$$

Proposition 1 then implies that a recursive equilibrium exists if there are functions $\bar{M} : \mathbf{S} \to \mathbb{R}^H_+$, $\bar{M} \in \widetilde{\mathbf{M}}$, such that for each $s = (z, \kappa) \in \mathbf{S}$ there are choices $(\bar{c}_h, \bar{\alpha}_h)$ for each agent $h \in \mathbf{H}$ with $\bar{\alpha} \in \mathbf{\Xi}$, $\bar{M}_h(s) = u'_h(z, \bar{c}_h)$ and

$$\begin{aligned} (\bar{c}_h, \bar{\alpha}_h) \quad \in \quad & \arg \max_{c_h \in \mathbb{R}_{++}, \alpha_h \in \mathbb{R}_+} E^{\bar{M}}_h(s, c_h, \alpha_h, \bar{\alpha}) \text{ s.t.} \\ & c - \kappa_h + \alpha_h \leqslant 0. \end{aligned} \tag{19}$$

As above the exogenous transition probability $\mathbb{P}$ implies a transition probability $\mathbb{Q}(.|s, \alpha)$ on $\mathcal{S}$. Under Assumption 4.1, 4.2 and Assumption 11, Lemma 1 holds as in Section 3. We define

$$\mathbf{P}_{\bar{M}}(s) = \left\{ \left(u'_h(z, \bar{c}_h)\right)_{h \in \mathbf{H}} : \exists \bar{\alpha} \in \mathbf{\Xi} \text{ such that } (\bar{c}_h, \bar{\alpha}_h) \text{ solves (19) for each } h \right\}.$$

While this correspondence is not guaranteed to be non-empty valued, we can define $\mathbf{P}^{co}_{\bar{M}}$ by requiring $\mathbf{P}^{co}_{\bar{M}}(s) = \text{conv}\left(\mathbf{P}_{\bar{M}}(s)\right)$ for all $s \in \mathbf{S}$. Let $\mathbf{R}(\bar{M})$ be the set of (equivalence classes of) measurable selections of $\mathbf{P}_{\bar{M}}$, and $\mathbf{R}^{co}(\bar{M})$ the set of measurable selections of $\mathbf{P}^{co}_{\bar{M}}$. Note that for any $\mathbf{M} \subset \widetilde{\mathbf{M}}$ this defines a (possibly empty-valued) correspondence $\mathbf{R}^{co} : \mathbf{M} \rightrightarrows L^m_\infty$. As in Lemma 3 for each $\bar{M} \in \widetilde{\mathbf{M}}$, the correspondence $\mathbf{P}_{\bar{M}}(s)$ is measurable.

[12] In this setting, we could equivalently use agents' investment policies instead of marginal utilities as the unknown functions. However, in the general setting above, with several goods and several types of capital this is no longer an option.

To show that there exists a (non-empty) convex and weak* compact set $\mathbf{M}^* \subset \widetilde{\mathbf{M}}$ such that $\mathbf{R}^{co}(\bar{M})$ is non-empty and $\mathbf{R}^{co}(\bar{M}) \subset \mathbf{M}^*$ for all $\bar{M} \in \mathbf{M}^*$, define a weak* compact and convex set

$$\mathbf{M}^0 = \{M \in L^m_\infty : \frac{1}{2}\underline{m} \leqslant M_h \leqslant 2\bar{m} \ a.s. \text{ for all } h \in \mathbf{H}\}.$$

For any set $\mathbf{M} \subset \widetilde{\mathbf{M}}$ define a correspondence $s \rightrightarrows \mathbf{P}^{co}_{\mathbf{M}}(s)$ by

$$\mathbf{P}^{co}_{\mathbf{M}}(s) = \operatorname{conv}\left(\bigcup_{M \in \mathbf{M}} \mathbf{P}_M(s)\right),$$

and denote by $\mathbf{R}^{co}(\mathbf{M})$ the set of measurable selections of $\mathbf{P}^{co}_{\mathbf{M}}$. We construct the set $\mathbf{M}^*$ inductively by defining for $i = 0, 1, 2 \ldots$, $\mathbf{M}^{i+1} = \mathbf{R}^{co}(\mathbf{M}^i) \cap \mathbf{M}^0$. Note that by the construction of $\mathbf{M}^i$ and by Lemma 4 any element of $\mathbf{M}^i$ consists of equilibrium marginal utilities for an artificial $(i+1)$-period economy where in the last period agents have some bounded continuation utility $M^0 \in \mathbf{M}^0$. Defining for each $h = 1, \ldots, H$, $M^0_h = u'_h(z, \kappa_h)$, it follows from Assumption 10 that $\mathbf{P}_{M^0}(s)$ is non-empty for every s and therefore, using Lemma 4, it follows that $\mathbf{R}^{co}(M^0)$ is non-empty. Moreover, it is clear from the construction of $\mathbf{M}^0$ that whenever $M^1 \in \mathbf{R}^{co}(M^0)$ we must have $M^1 \in \mathbf{M}^0$. Therefore $\mathbf{M}^1$ is non-empty. From the same argument it follows that there is some $M^2 \in \mathbf{R}^{co}(M^1)$ with $M^2 \in \mathbf{M}^0$ and recursively that for each i there is $M^i \in \mathbf{R}^{co}(M^{i-1})$ with $M^i \in \mathbf{M}^0$. Therefore each $\mathbf{M}^i$ is non-empty. Obviously each $\mathbf{M}^i$ is also convex. By the same argument as in the proof of Lemma 3 each $\mathbf{M}^i$ is weak* compact. Obviously we have $\mathbf{M}^1 \subset \mathbf{M}_0$ and if $\mathbf{M}^i \subset \mathbf{M}^{i-1}$ it must follow that $\mathbf{M}^{i+1} \subset \mathbf{M}^i$. Therefore we must have for any $i = 0, 1, \ldots$ that for all $M \in \mathbf{M}^i$

$$\mathbf{R}^{co}(M) \cap \mathbf{M}^0 \subset \mathbf{M}^i.$$

Assumptions 8–10, together with discounting guarantee that for T sufficiently large, $\mathbf{R}^{co}(M)$ must be non-empty for all $M \in \mathbf{M}^T$ and if $\bar{M} \in \mathbf{R}^{co}(M)$ then $\bar{M} \in \mathbf{M}_0$. Therefore we can take $\mathbf{M}^* = \mathbf{M}^T$. Having constructed a set $\mathbf{M}^*$ on which $\mathbf{P}_M(s)$ is non-empty valued for all s the rest of the proof is identical to the proof of Proposition 2. □

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